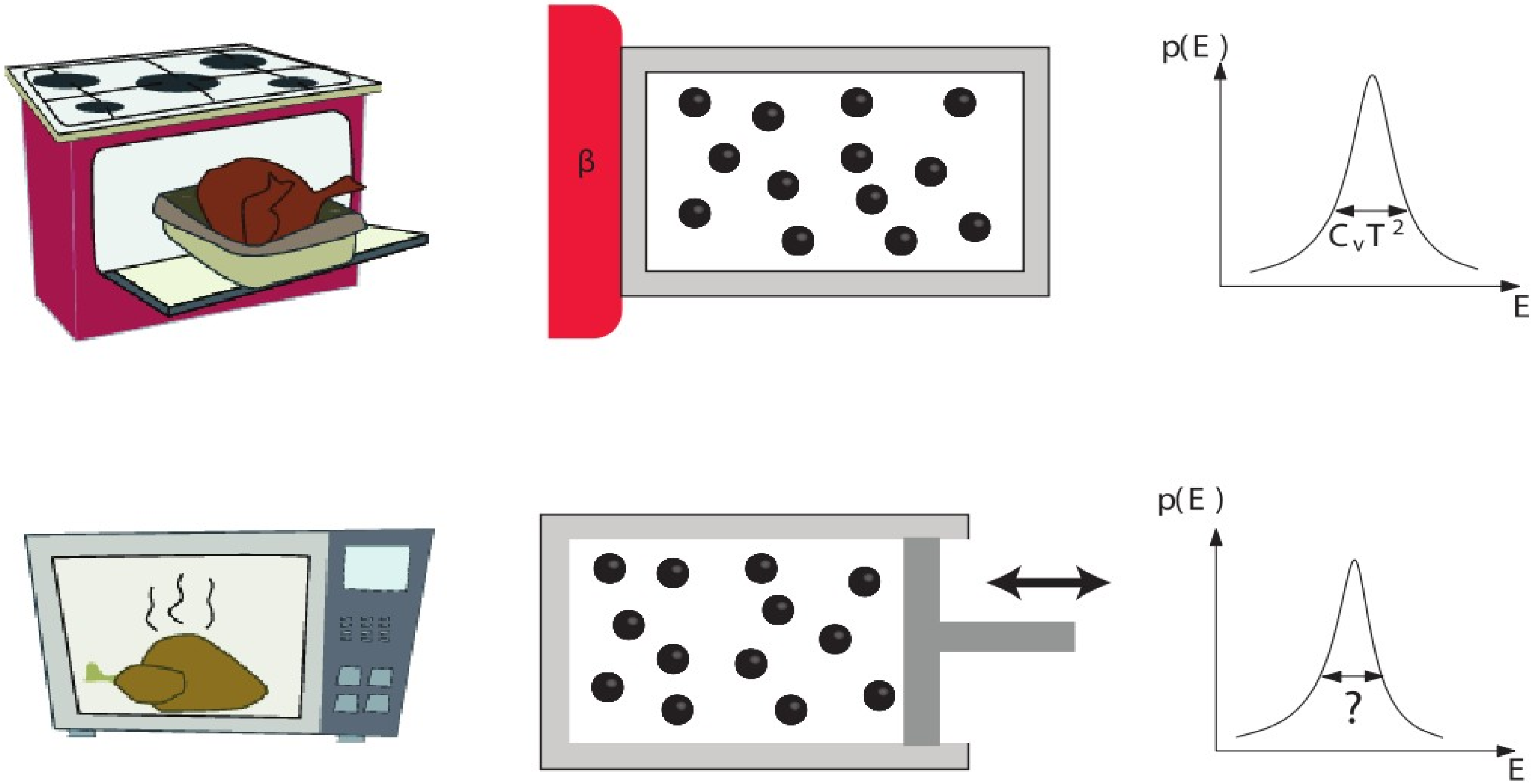
Universal energy fluctuation in thermally isolated driven systems

Cold atoms are almost perfectly isolated systems:

1. Probe coherent non-equilibrium dynamics for "long" times
2. Investigate foundations of statistical mechanics

Luca D'Alessio (BU), Anatoli Polkovnikov (BU),
Yariv Kafri (Technion), Guy Bunin (Technion), Paul Krapivsky (BU)

Thermal Vs Not-Adiabatic Heating



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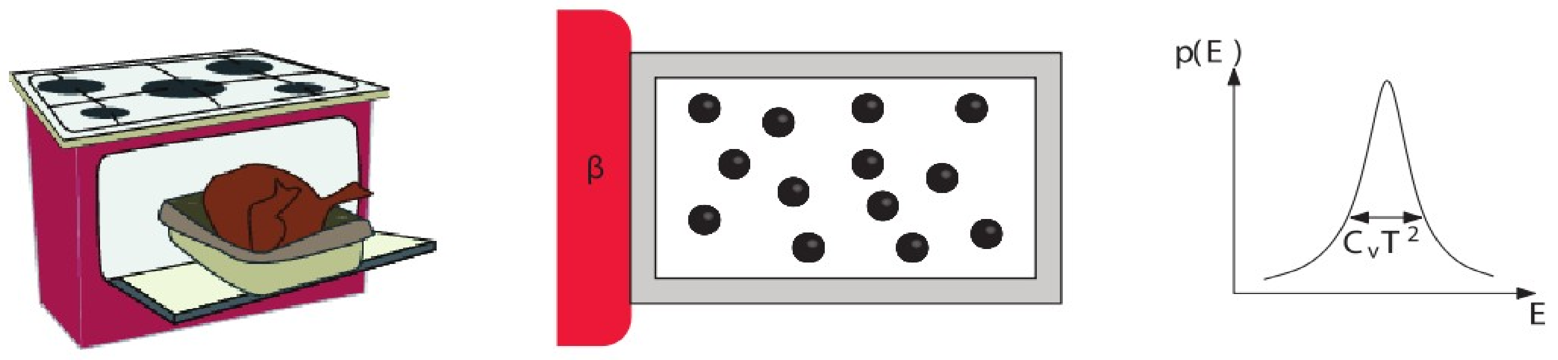

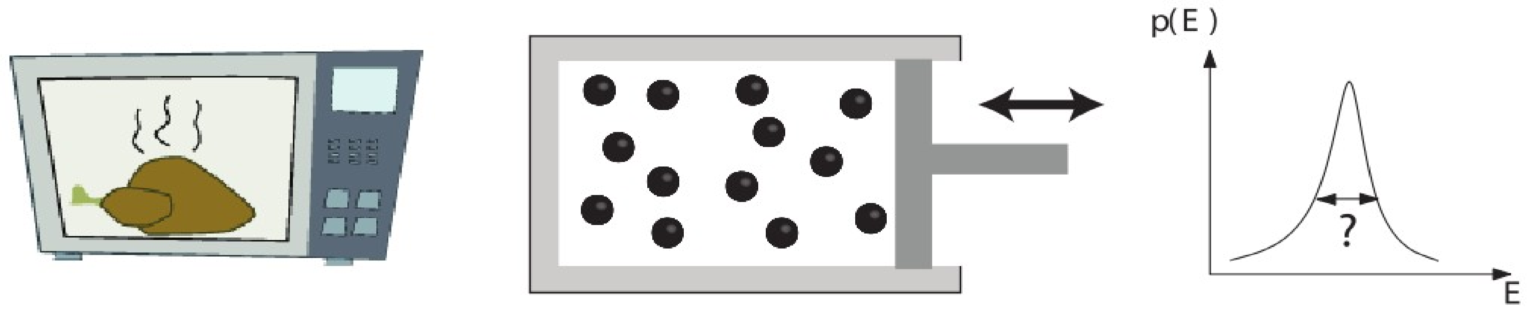

- Can we increase the energy without increasing the **uncertainty** in its final value?
- Does the energy distribution look like a thermal energy distribution at some **effective temperature**?

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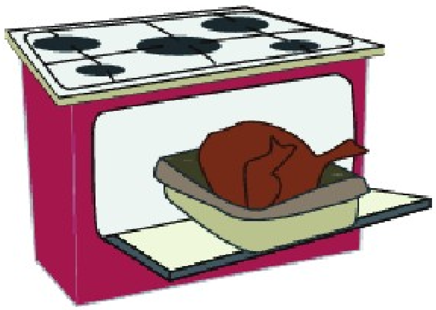

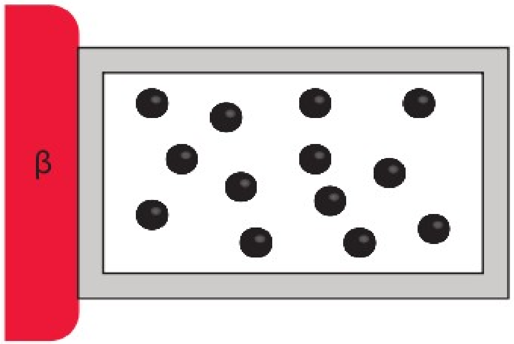

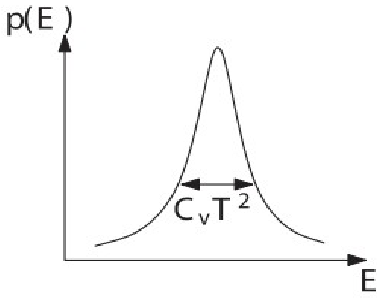

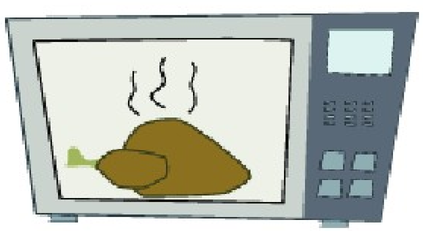

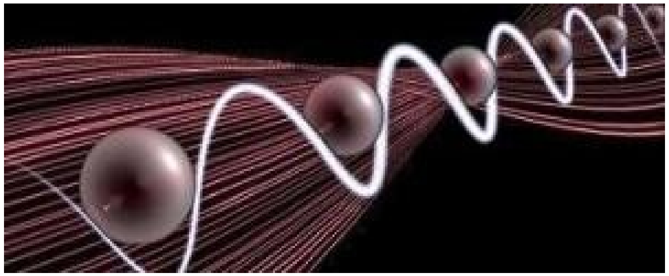

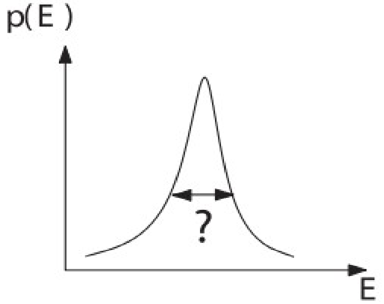

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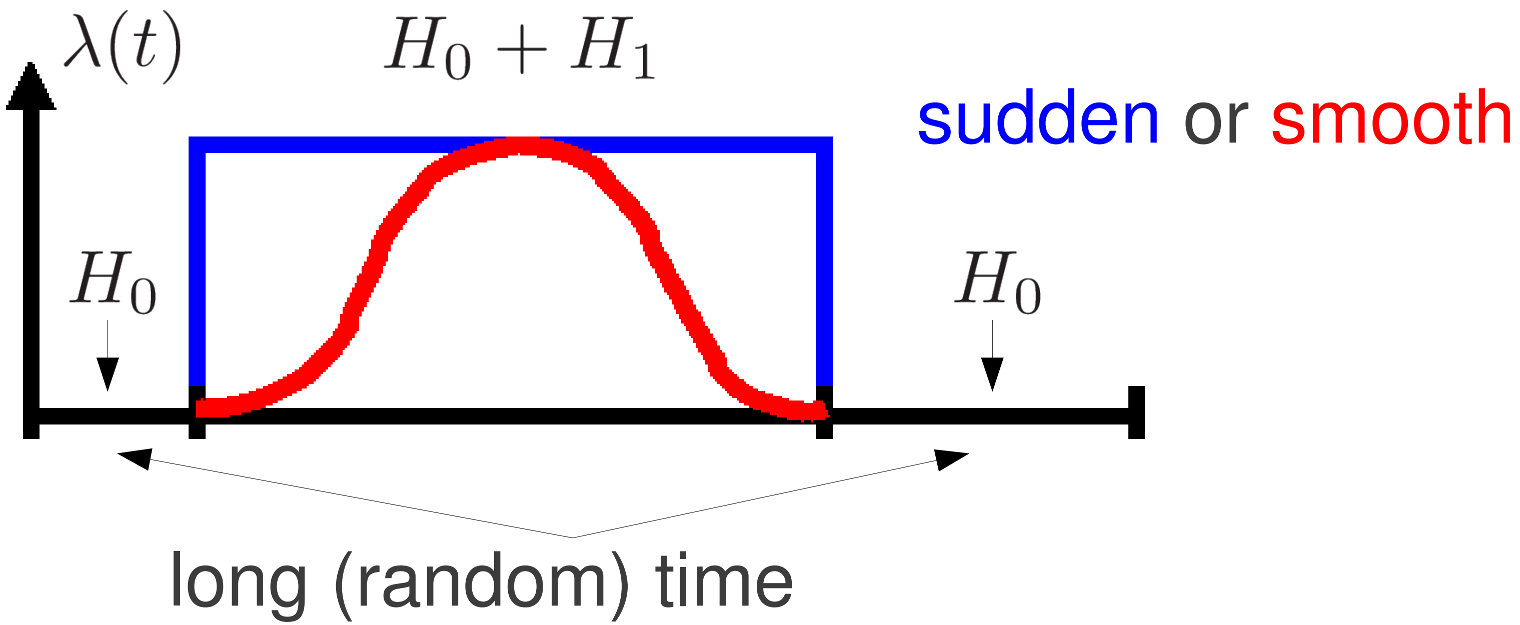
Setup: $H(t) = H_0 + \lambda(t)\, H_1$



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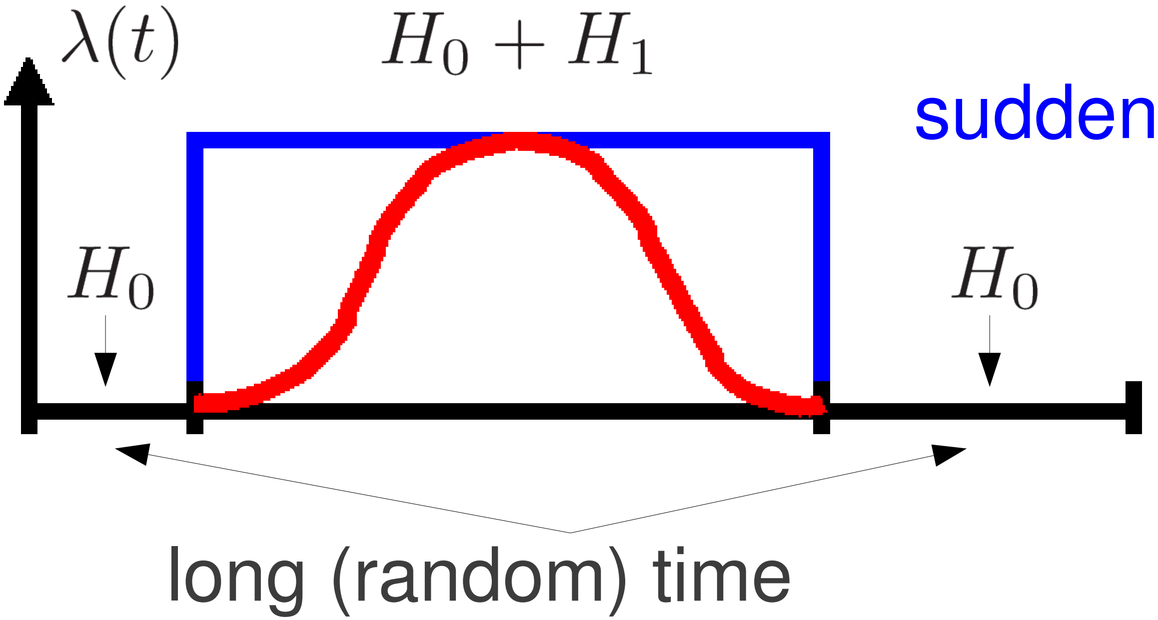

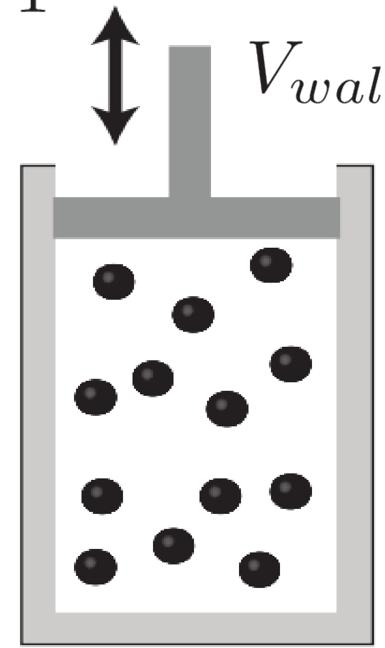

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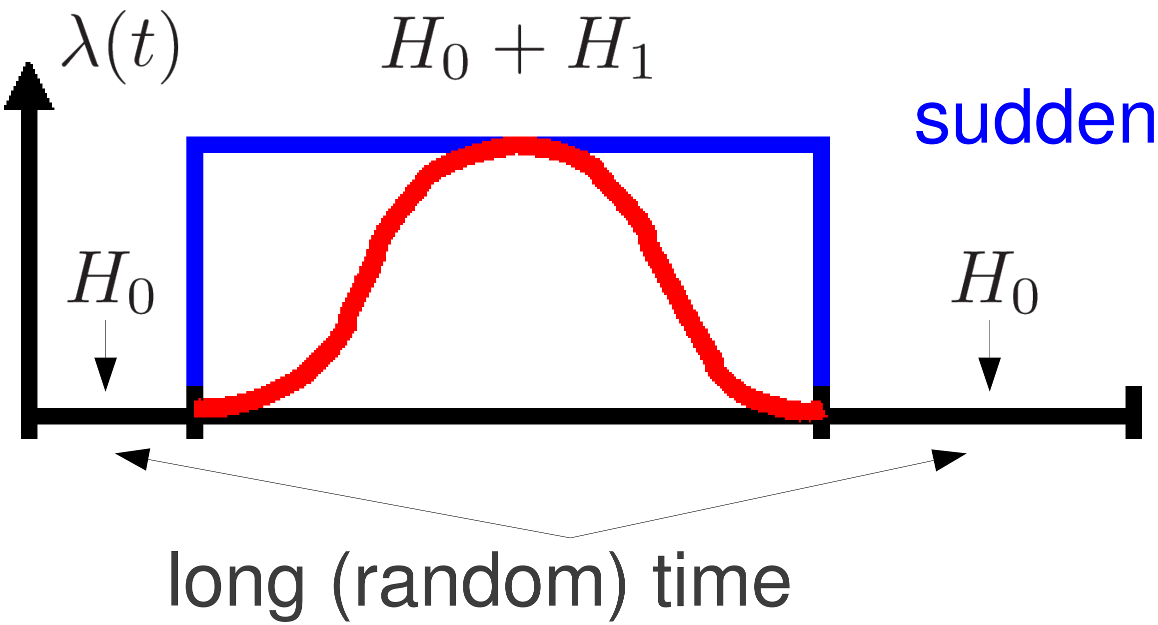
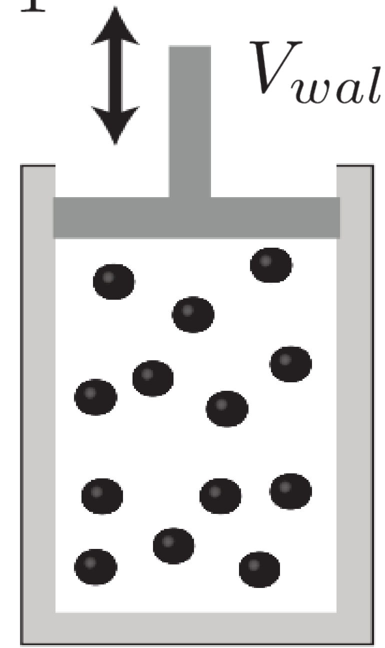


sudden or smooth



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Not Adiabatic: $E_f^{gas} > E_i^{gas}$ and E_f^{gas} is a random variable

(random waiting times, initial energy of the gas does not fix the individual particles' positions and velocities)

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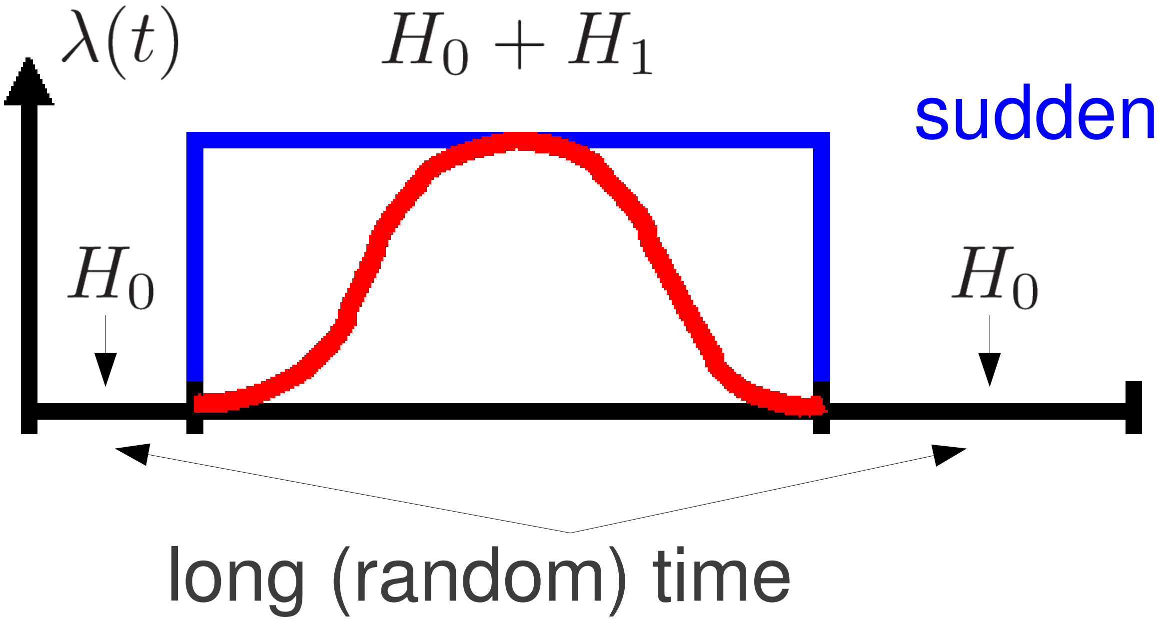

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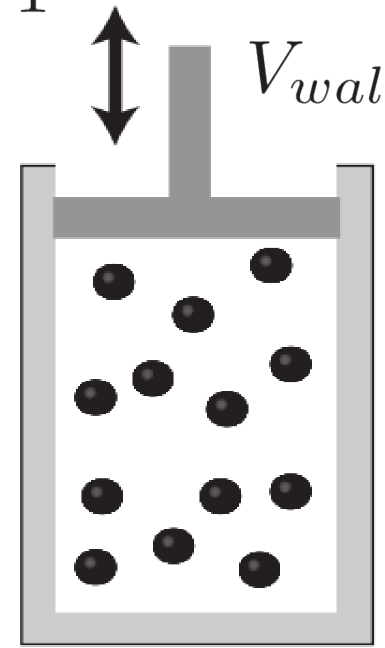

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ENSEMBLE AVERAGE over protocols and initial conditions

Isolated System = Unitary Evolution

Von Neumann equation (quantum Liouville's theorem): $$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar}[H(t);\rho(t)]$$

is GENERAL, EXACT but VERY VERY HARD.

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Outline

1. What I have learned about unitary evolution
2. Application to repeated quenches problem
3. Appendix

Unitary Evolution

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$$\rho_{nn} = \sum_m U_{nm} \rho_{mm} U^\dagger_{mn} = \sum_m |U_{nm}|^2 \rho_{mm} \equiv \sum_m T_{mn} \rho_{mm}$$

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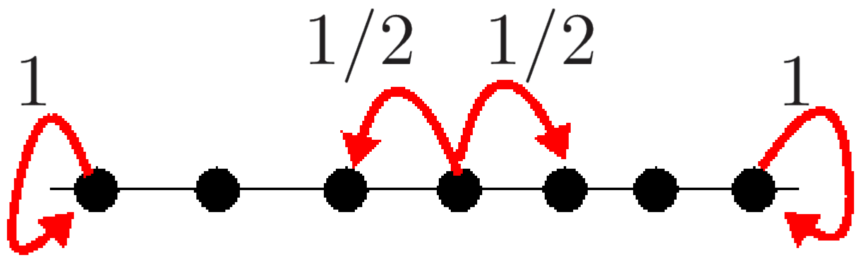
1st advertisement: Allahverdyan et al, EPL **95** (2011) 60004

Work extraction from a microcanonical bath

Double Stochasticity: $\sum_m T_{nm} = \sum_n T_{mn} = 1$

NO



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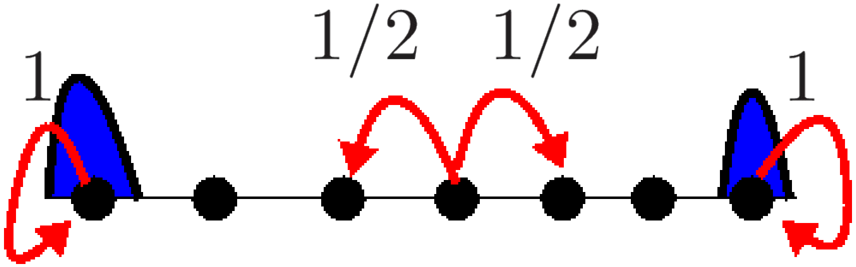

No absorbing states

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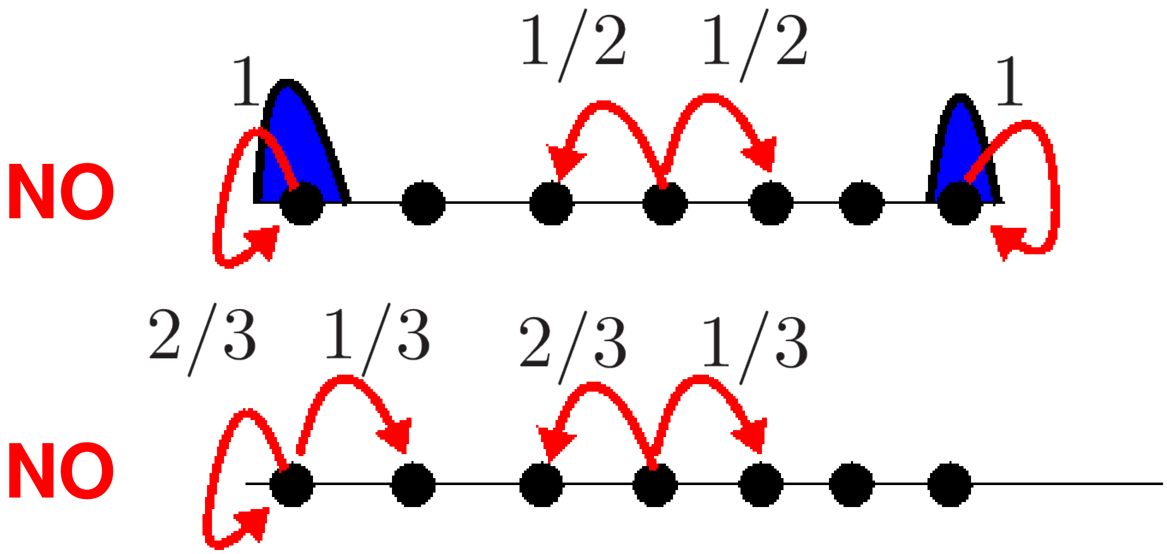

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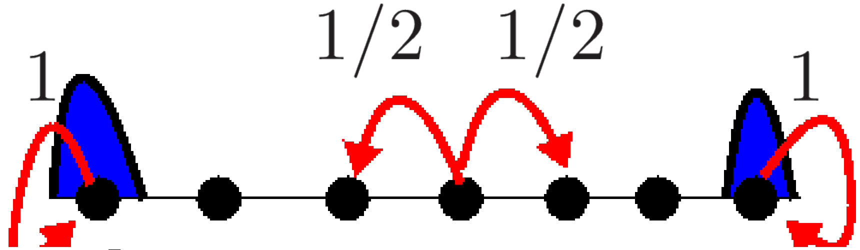

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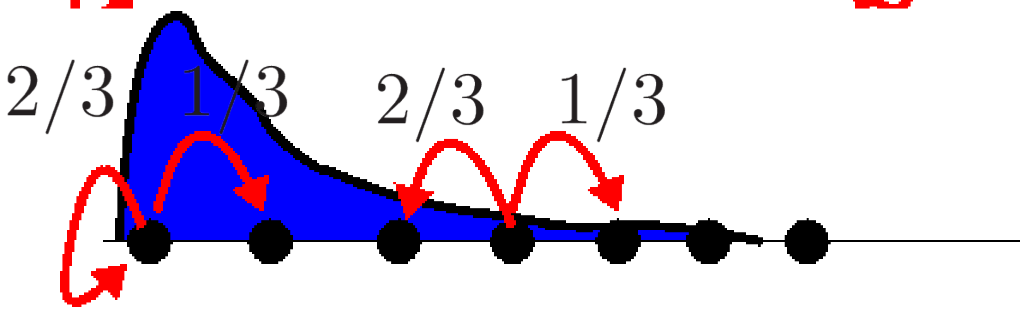

No probability localization close GS

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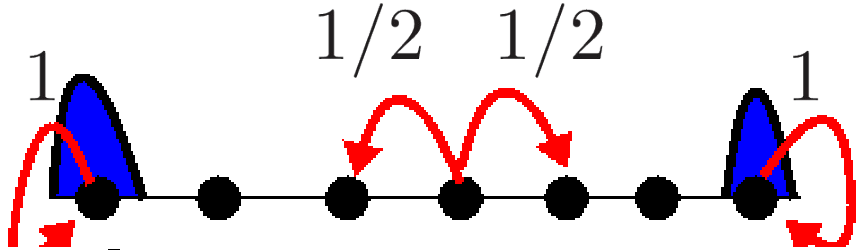

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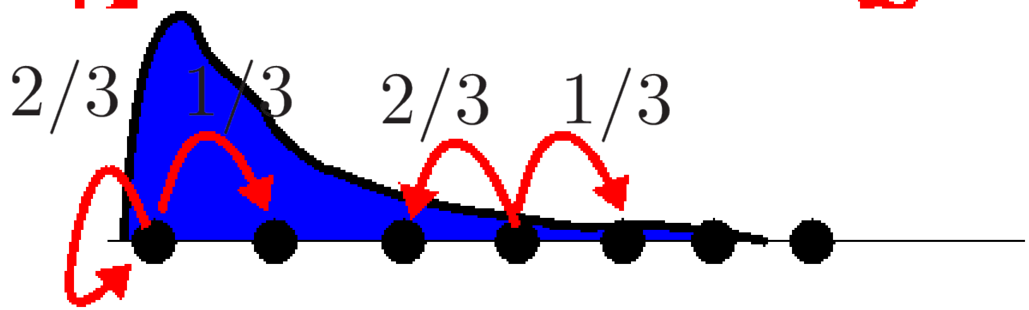

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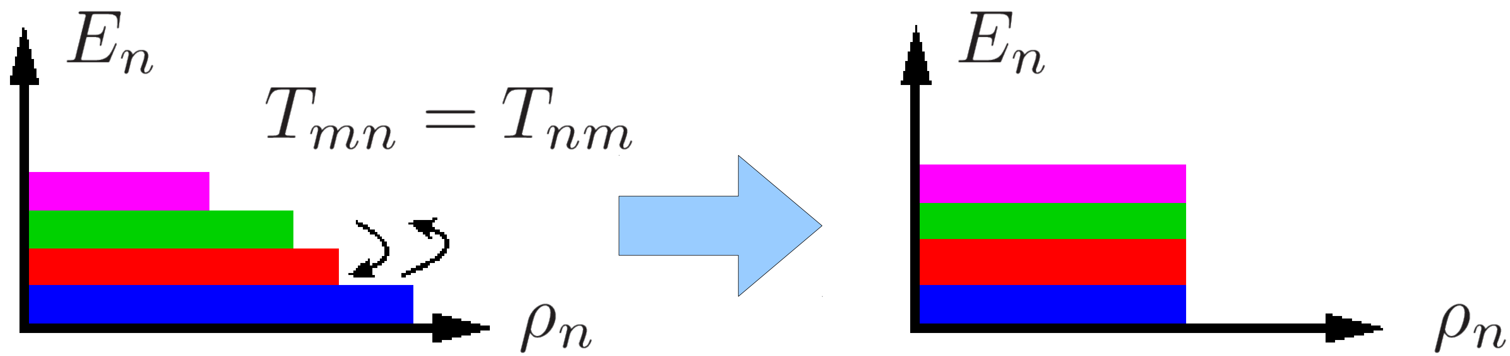

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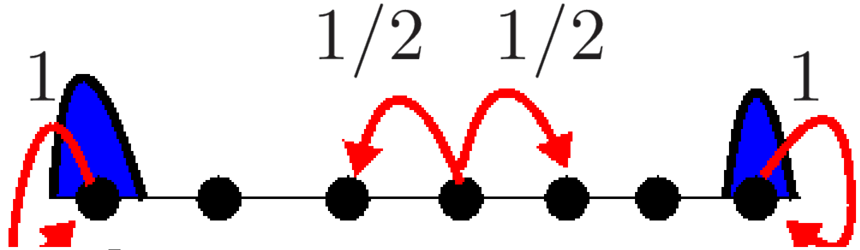

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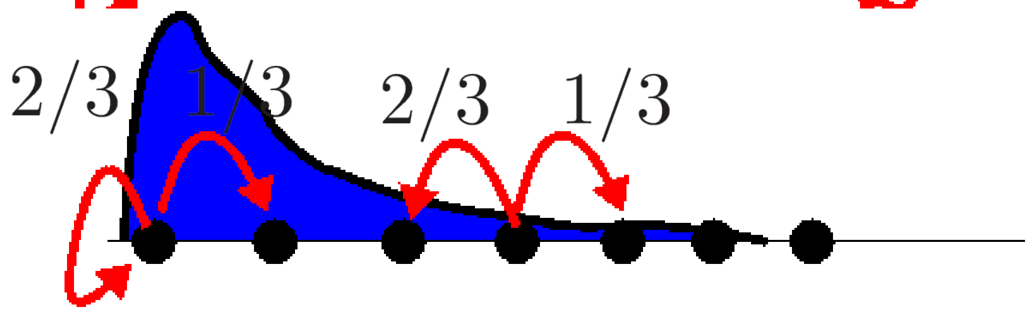

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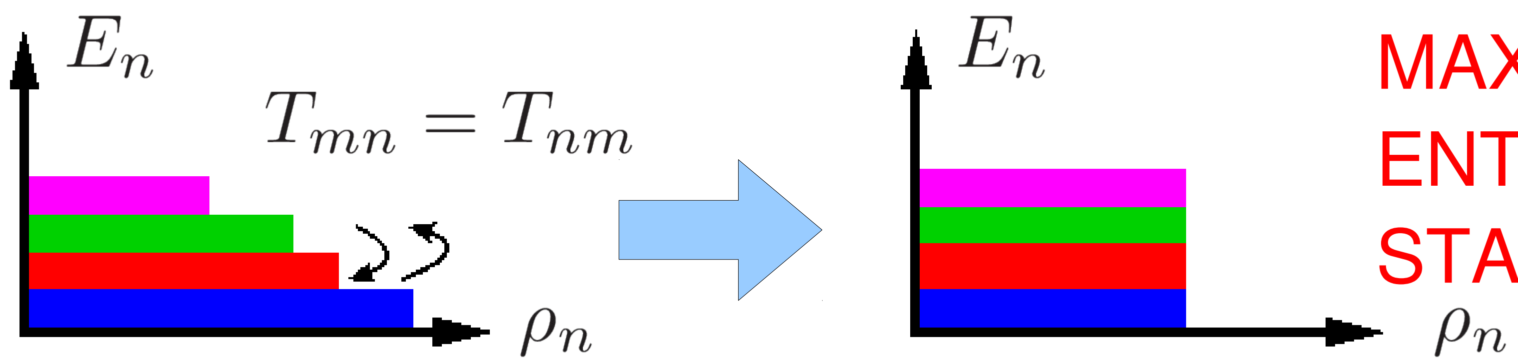

MAXIMUM ENTROPY STATE

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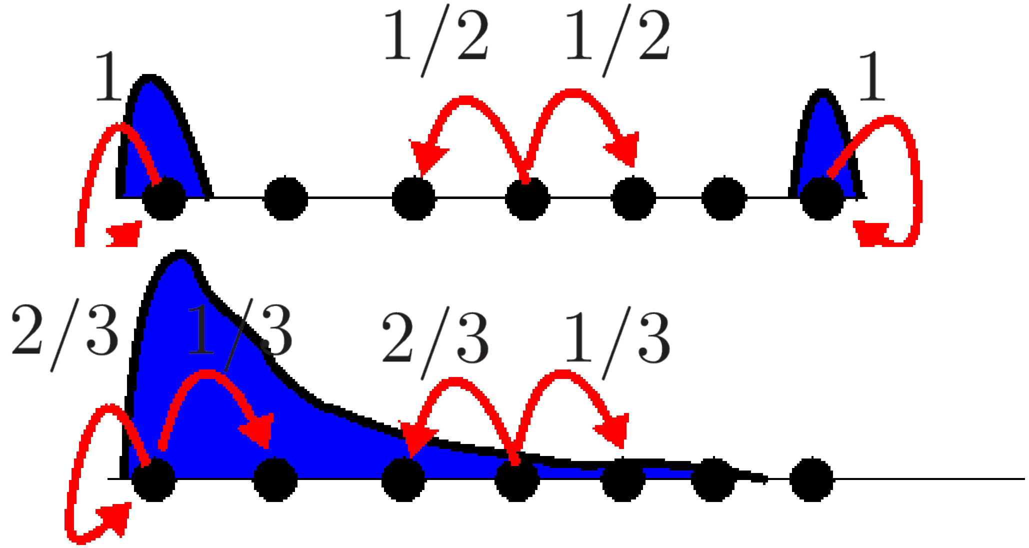

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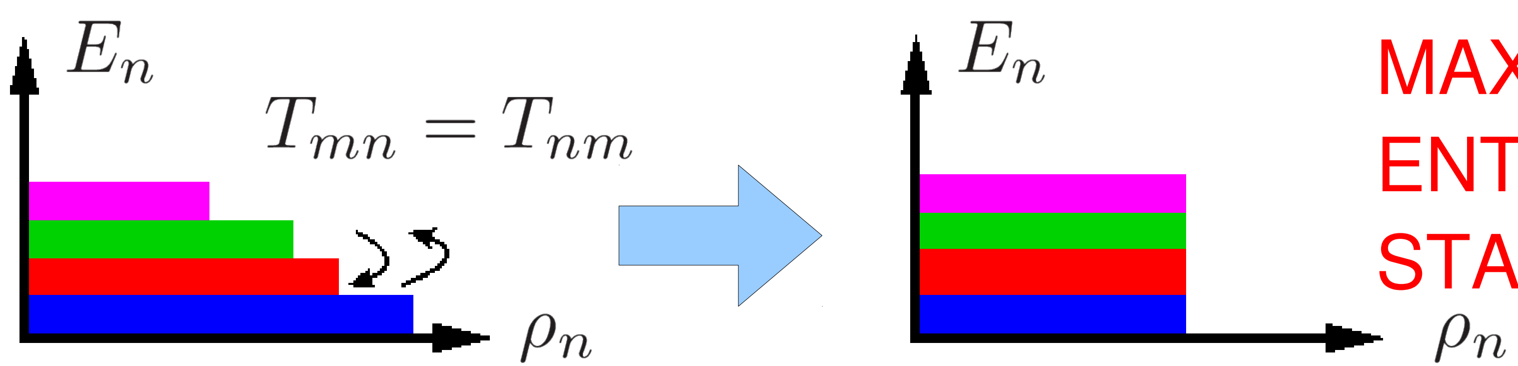


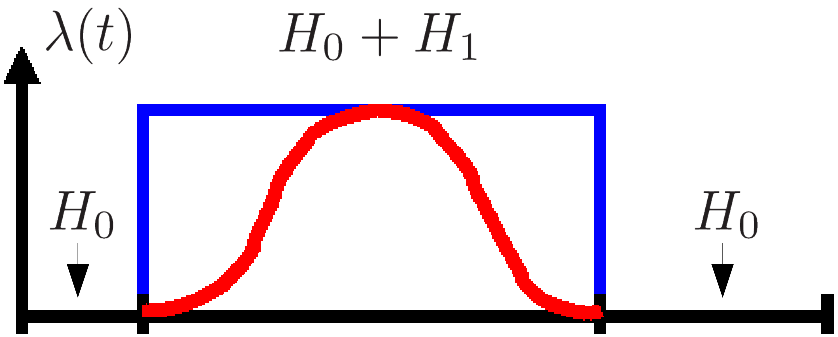
MAXIMUM ENTROPY STATE

First take-home message:

- Unitary evolution tends to bring you towards a maximum Shannon (diagonal) entropy state: $S_{sh} = -\sum_n \rho_n \, log \, \rho_n$
- This is 2nd law of thermodynamics

A. Polkovnikov Annals Phys **326**, 486 (2011)

Off-diagonal elements



Write ρ in the base of H_0

During the waiting times diagonal elements are fixed while off-diagonal ones oscillate

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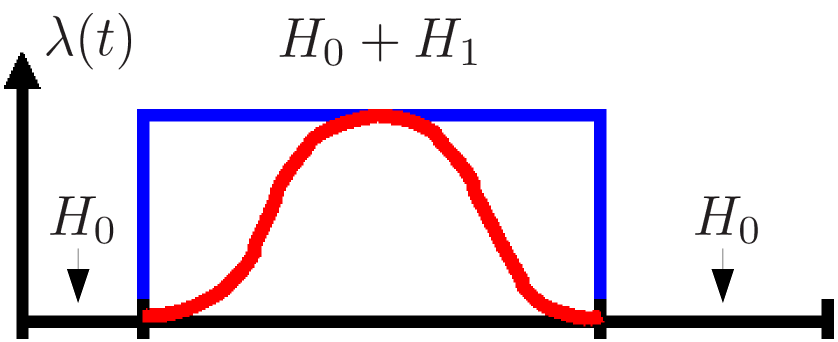

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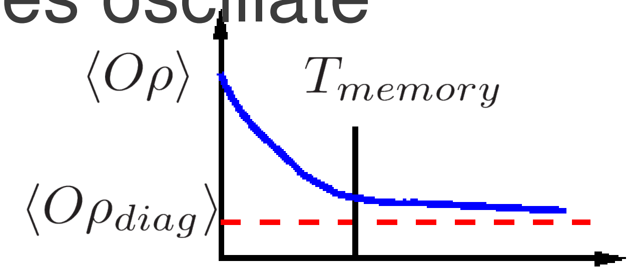
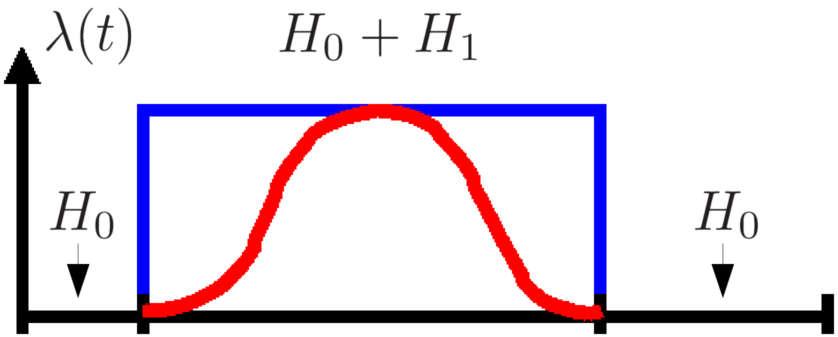


Time after quench

Memory is encoded in off-diagonal $\rightarrow$ for chaotic system take: $T_{wait} \geq T_{memory}$

Alternatives to ETH, by Rigol et al.

Off-diagonal elements



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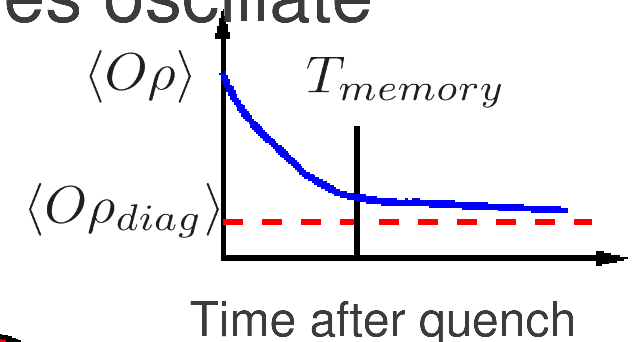

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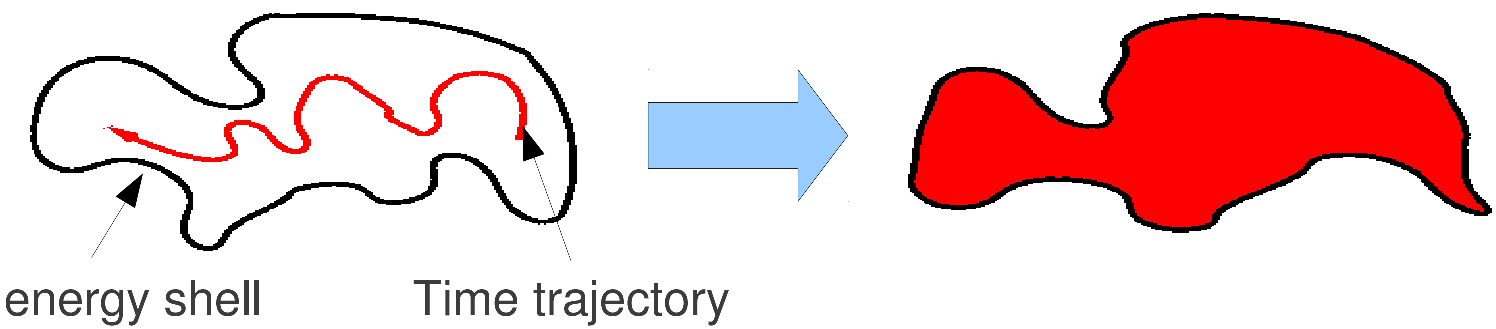

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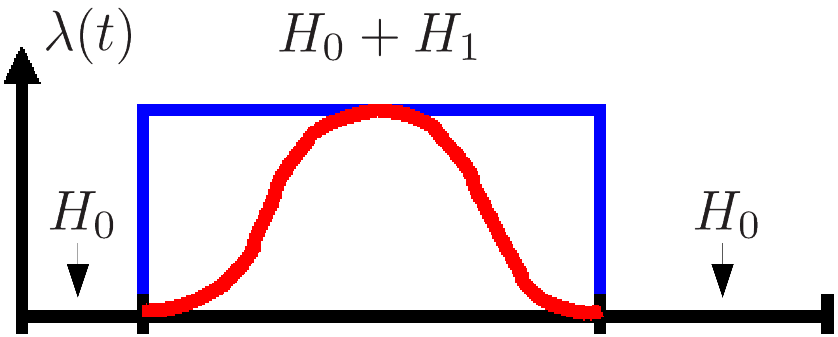

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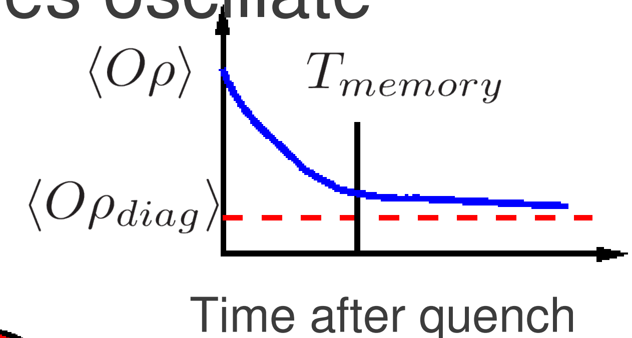

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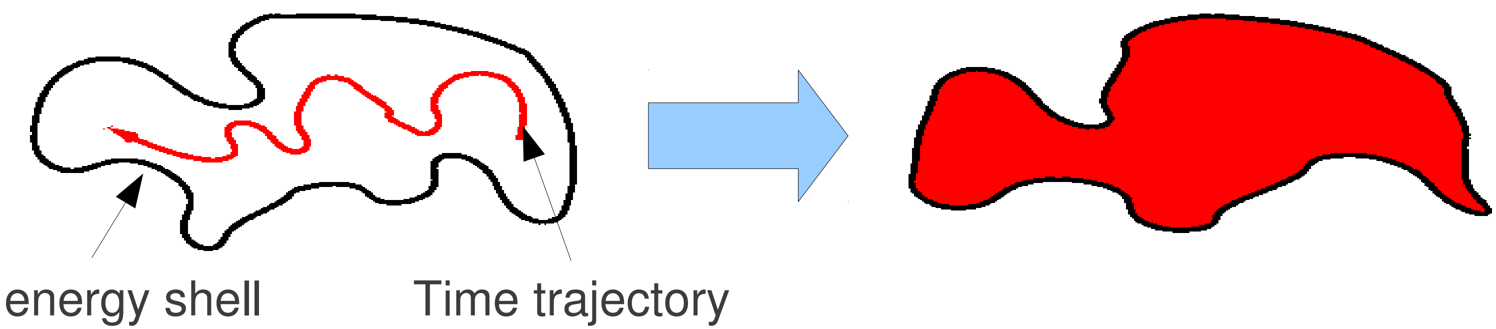

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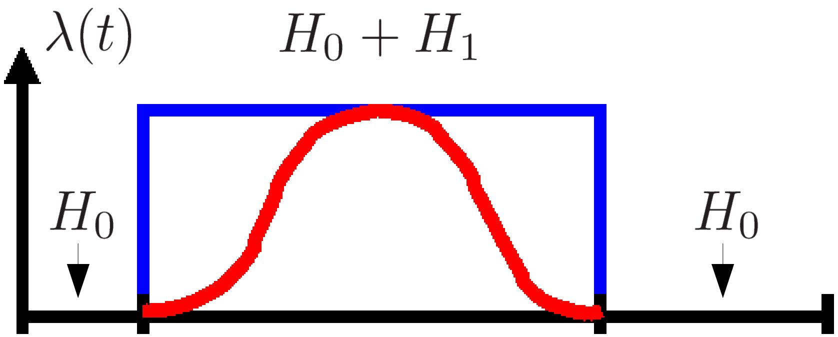
Gedankenexperiment:

Sequence of spins polarized (at random) in xy plane.

Is there any way to distinguish that from the diagonal ensemble?

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Off-diagonal elements



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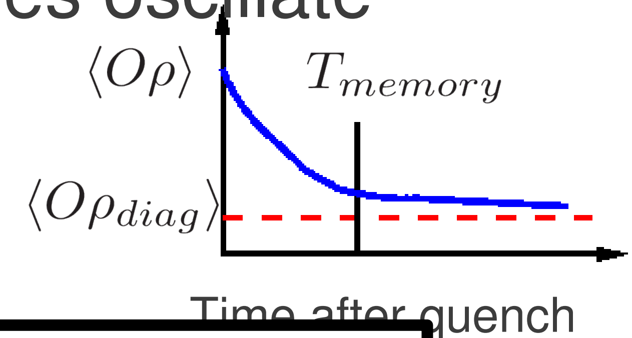

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Restart each cycle from diagonal ensemble

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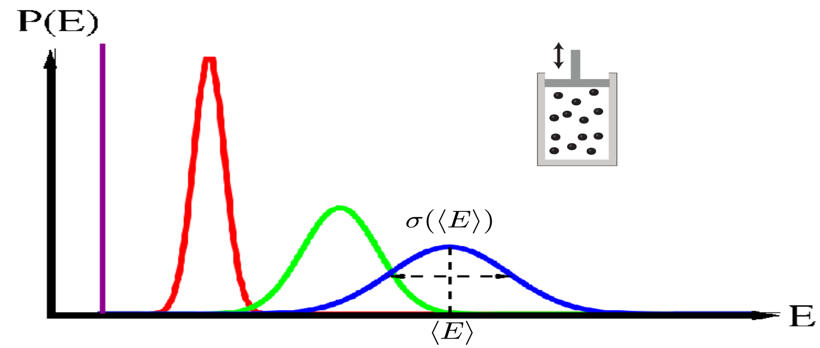
Off-diagonal elements

We assume that between the cycles the system reaches a steady state (or a diagonal ensemble [21] in the quantum language) so that its state is fully characterized by its energy distribution. In ergodic systems this requirement can be satisfied by waiting between cycles a time which is longer than the relaxation time of the system. In non ergodic (integrable) systems this can be achieved by having a long fluctuating time between cycles. This effectively leads to an additional time averaging which is equivalent to the assumption of starting from a diagonal ensemble. (For more details about relaxation to asymptotic states in integrable systems see Ref. [5] and refs. therein). To make this discussion more concrete consider, for example, a compression and expansion of the piston in Fig. 1 according to an arbitrary protocol. The gas is allowed to relax between the cycles (when the piston is stationary) at a fixed energy. For a weakly interacting ergodic gas such a relaxation implies that the momentum distribution of individual particles assumes a Maxwell-Boltzmann form together with a randomization of the coordinate distribution. For a noninteracting gas in a chaotic cavity the relaxation implies conservation of the individual energies of each particle and a randomization of the coordinates and directions of their motion. And finally for noninteracting particles in a regular non-chaotic cavity the relaxation implies a randomization of the coordinates within individual periodic trajectories. Therefore, in the beginning of each cycle there are no correlations between positions and velocities of particles within the available phase space.

Master Equation



$$\rho_n = \sum_m T_{mn} \rho_m, \quad \sum_m T_{nm} = \sum_n T_{nm} = 1$$

$$P(E) = \Omega(E)\rho(E)$$

Algebra + Technical Reasons (appendix1)

$$P(E, t+dt) = \int_{-\infty}^{\infty} dW \, T_{E-W}(W) \, P(E-W, t), \quad \Omega(E) T_{E \to E'} = \Omega(E') \widetilde{T}_{E' \to E}$$

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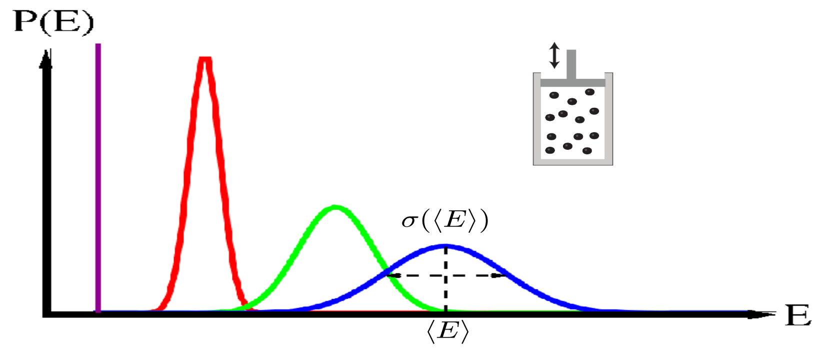

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Exact if you restart each cycle from the diagonal ensemble

QM is still encoded in the Transition rates (appendix2)

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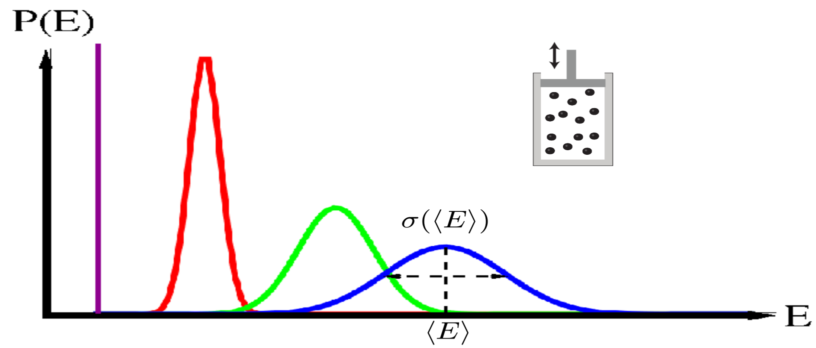

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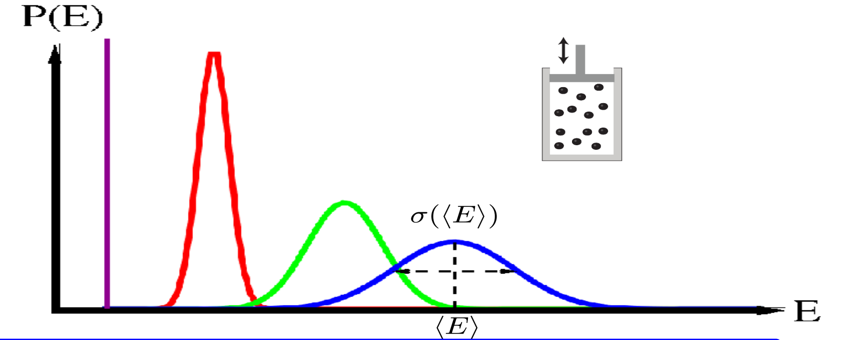
Expand (our goal is to calculate $\sigma(\langle E\rangle)$)

$$\partial_t P(E) = -\partial_E(A(E)P(E)) + \frac{1}{2}\partial_{EE}(B(E)P(E)) + \ldots., \quad 2A(E) = \beta(E)B(E) + \partial_E B(E)$$

where: $\beta(E) = \partial_E S(E)$ is the microcanonical temperature. This is "generalized Einstein relation" between drift and diffusion in open systems $\rightarrow$ Jarzynski equality (appendix3)

Master Equation



Second take-home message:

Unitary evolution can be approximated by a Fokker–Planck equation where drift and diffusion are constrained **a priori**

Second advertisement:

“Energy diffusion in a chaotic adiabatic billiard gas”.
C. Jarzynski, Phys. Rev. E **48**, 4340–4350 (1993)

“Thermalisation of a closed quantum system:
From many-body dynamics to a Fokker-Planck equation”
C. Ates, J. P. Garrahan, I. Lesanovsky, arXiv:1108.0270

Solve the Fokker–Planck equation

We turn the Fokker-Plank equation into a relation between the first and second moments (by integration by parts)

$$\frac{\partial \sigma^2}{\partial \langle E \rangle} = \frac{\langle B \rangle + 2(\,\langle A\, E \rangle - \langle A \rangle \langle E \rangle\,)}{\langle A \rangle}$$

Evaluate these averages using saddle-point approximation (narrow P(E) $\rightarrow$ mesoscopic systems)

$$\sigma^2(E) = \sigma_0^2 \frac{A^2(E)}{A^2(E_0)} + 2A^2(E) \int_{E_0}^{E} \frac{dE'}{A^2(E')\beta(E')}$$

Protocol **dependent** Protocol **independent**

Dynamical phase transition

$$\sigma^2(E) = \sigma_0^2 \frac{A^2(E)}{A^2(E_0)} + 2A^2(E) \int_{E_0}^{E} \frac{dE'}{A^2(E')\beta(E')}$$

Assume: $A(E) \sim E^s, \quad \beta(E) \sim E^{-\alpha}, \quad \sigma_0^2(E_0) = 0$

with: $s \leq 1$ (validity of FP) , $0 < \alpha \leq 1$ (Cv>0 and S(E) increasing unbounded function of energy)

As the energy increases ($E \to \infty$) the integral:

•Diverges if $2s - \alpha < 1 \to \sigma^2(E) \sim \dfrac{E}{\beta(E)}$ Protocol **independent**

•Converges if $2s - \alpha > 1 \to \sigma^2(E) \sim A(E)^2$ Protocol **dependent**

Dynamical phase transition

$$\sigma^2(E) = \sigma_0^2 \frac{A^2(E)}{A^2(E_0)} + 2A^2(E) \int_{E_0}^{E} \frac{dE'}{A^2(E')\beta(E')}$$

Define: $\eta = 2s - \alpha - 1$

•Diverges if $\eta < 0 \rightarrow \frac{\sigma^2(E)}{\sigma_{eq}^2(E)} \sim \frac{2\alpha}{|\eta|}$

Gibbs-like regime

•Converges if $\eta > 0 \rightarrow \frac{\sigma^2(E)}{\sigma_{eq}^2(E)} \sim \frac{2\alpha}{\eta} \left(\frac{E}{E_0}\right)^{\eta}$

Run-away regime

Diverging time scale: $\tau \sim \frac{1}{1-s} exp[\frac{1-s}{|\eta|}]$

"Continuous phase transition"

Results

•Can we increase the energy without increasing the **uncertainty** in its final value?

•Does the energy distribution look like a thermal energy distribution at some **effective temperature**?

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  ALMOST.

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  SOMETIMES.

Results

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  SOMETIMES.

$\frac{1-\alpha}{2}$ $\quad$ $\frac{1+\alpha}{2}$ $\quad$ $s=1$

$$\frac{\sigma^2(E)}{\sigma^2_{eq}(E)} < 1 \qquad \frac{\sigma^2(E)}{\sigma^2_{eq}(E)} > 1 \qquad \frac{\sigma^2(E)}{\sigma^2_{eq}(E)} \sim \frac{2\alpha}{\eta}\left(\frac{E}{E_0}\right)^{\eta}$$

Gibbs-like regime ($\eta < 0$) **Run-away** regime ($\eta > 0$)

$$Reminder: \quad s \leq 1, \;\; 0 < \alpha \leq 1, \;\; \eta = 2s - \alpha - 1$$

Conclusions

First take-home message: $\sum_m T_{nm} = \sum_n T_{mn} = 1$

- Unitary evolution tends to bring you towards a maximum Shannon (diagonal) entropy state: $S_{sh} = -\sum_n \rho_n \, log \, \rho_n$
- This is 2nd law of thermodynamics

A. Polkovnikov Annals Phys **326**, 486 (2011)

Second take-home message:

Unitary evolution can be approximated by a Fokker–Planck equation where drift and diffusion are constrained **a priori**

arXiv:1108.0270v1 [quant-ph]

3rd advertisement: Nature Physics doi:10.1038/nphys2057

Appendix 1: Master Equation

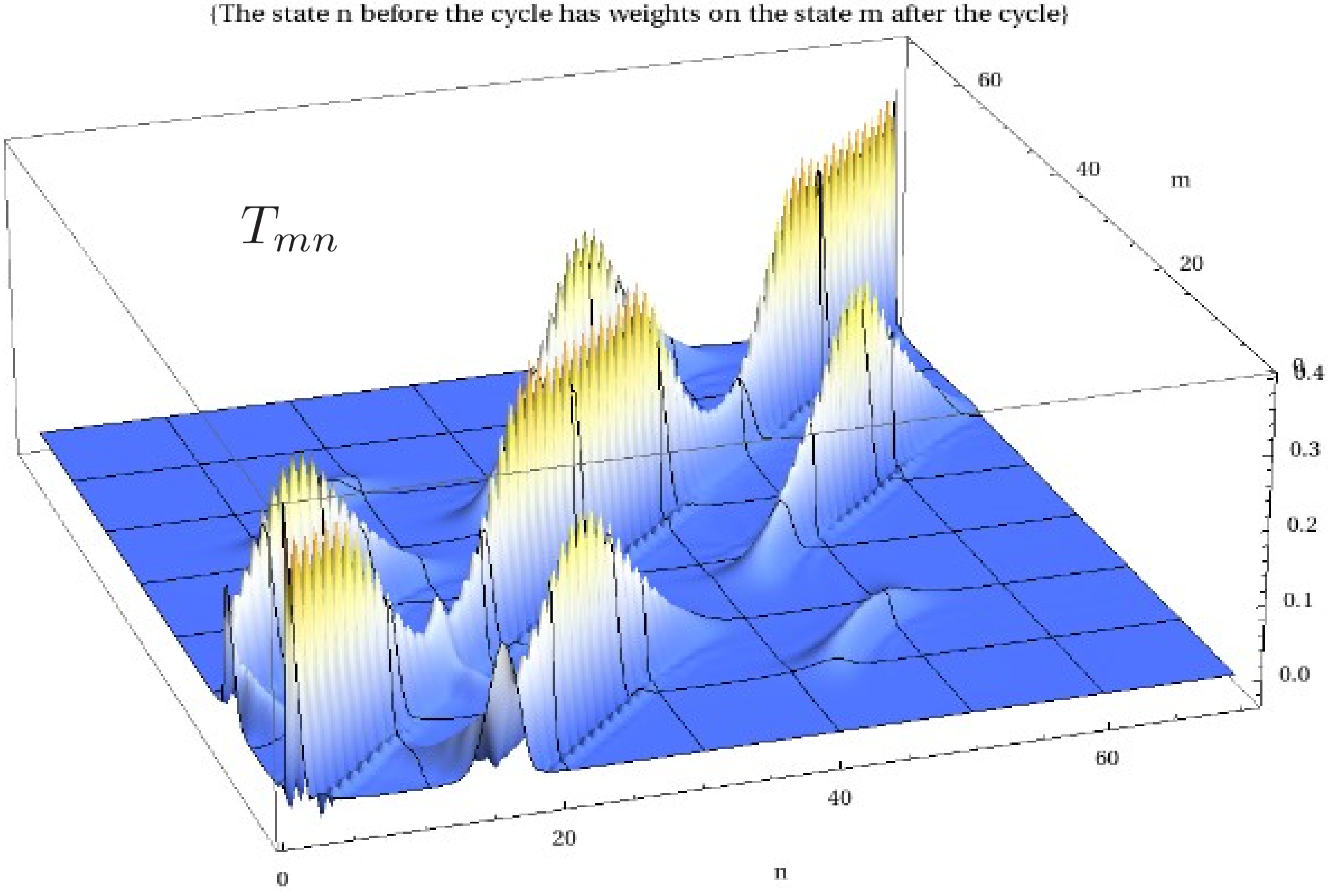
Appendix:

Distinguish chaotic from not chaotic Phys. Rev. Lett. 107, 040601 (2001)

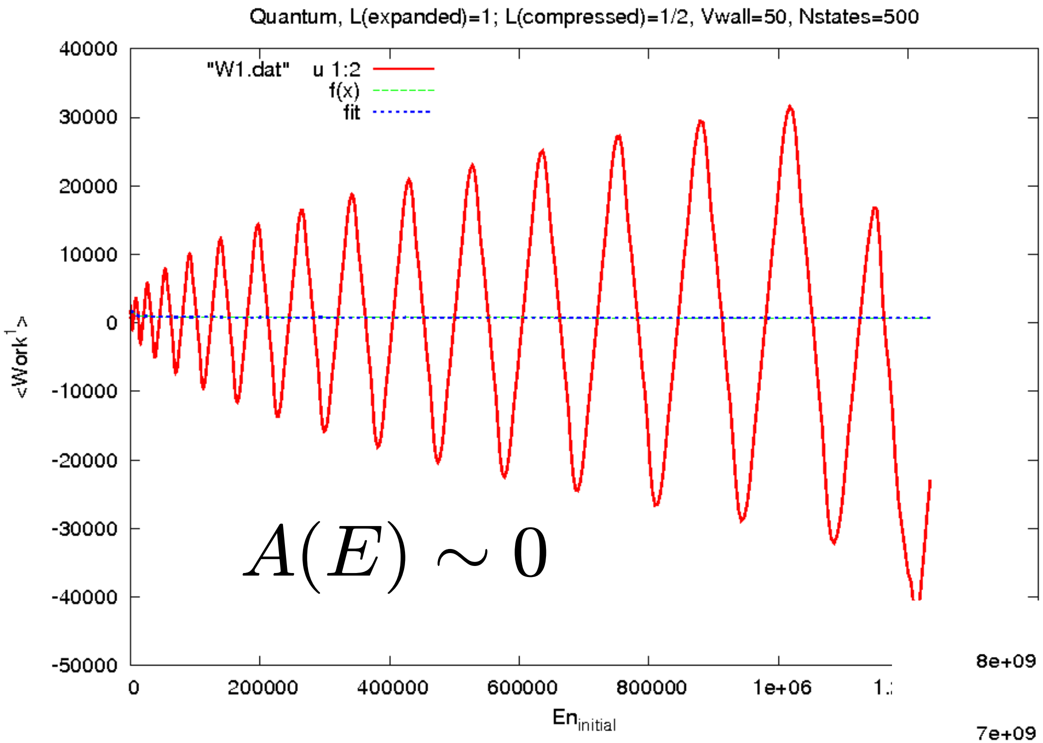
Is exp relevant <E>=int dE E P(E)

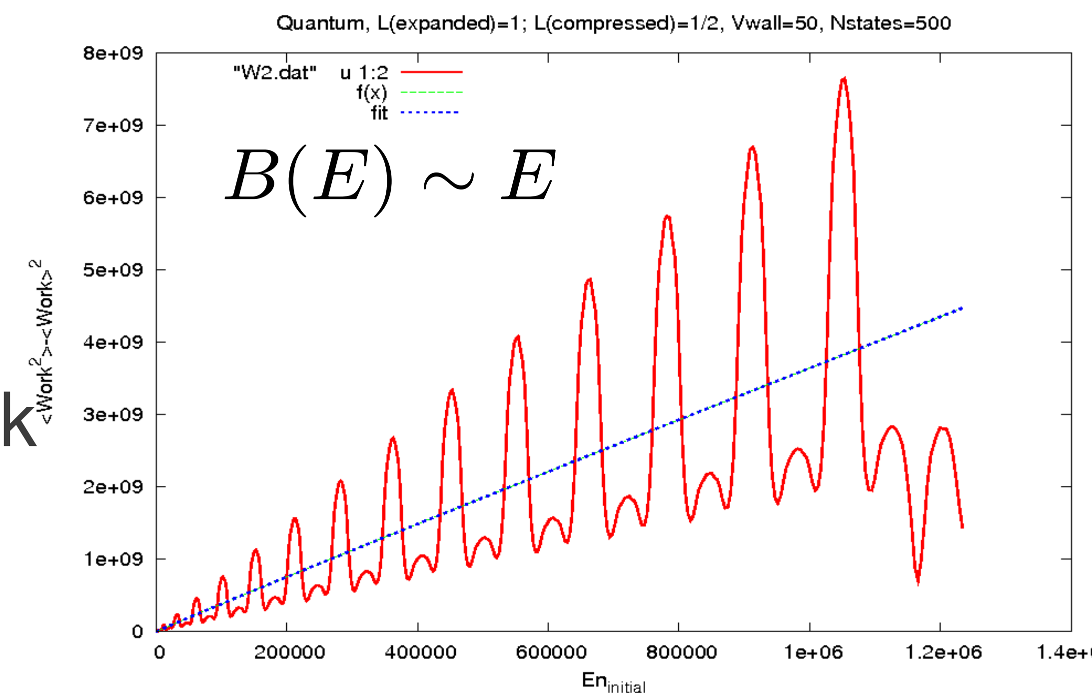
Makes my transition smooth

Appendix 2: linear quench in 1D quantum piston



1D quantum linear quench

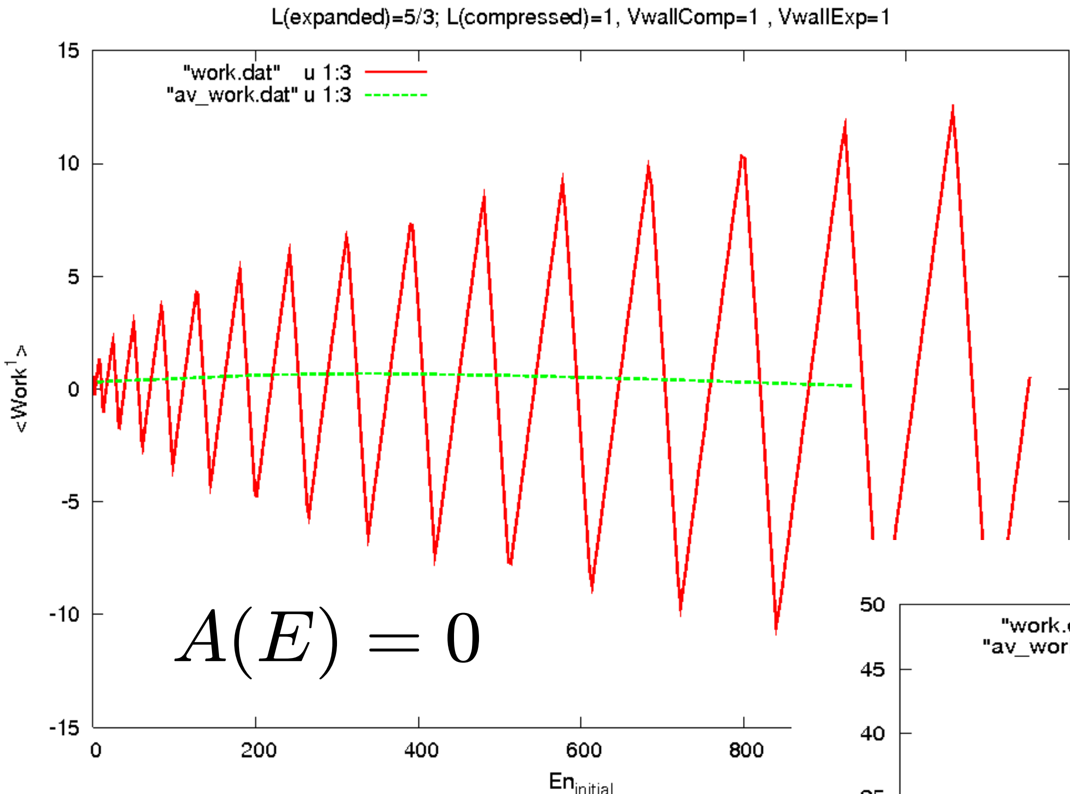




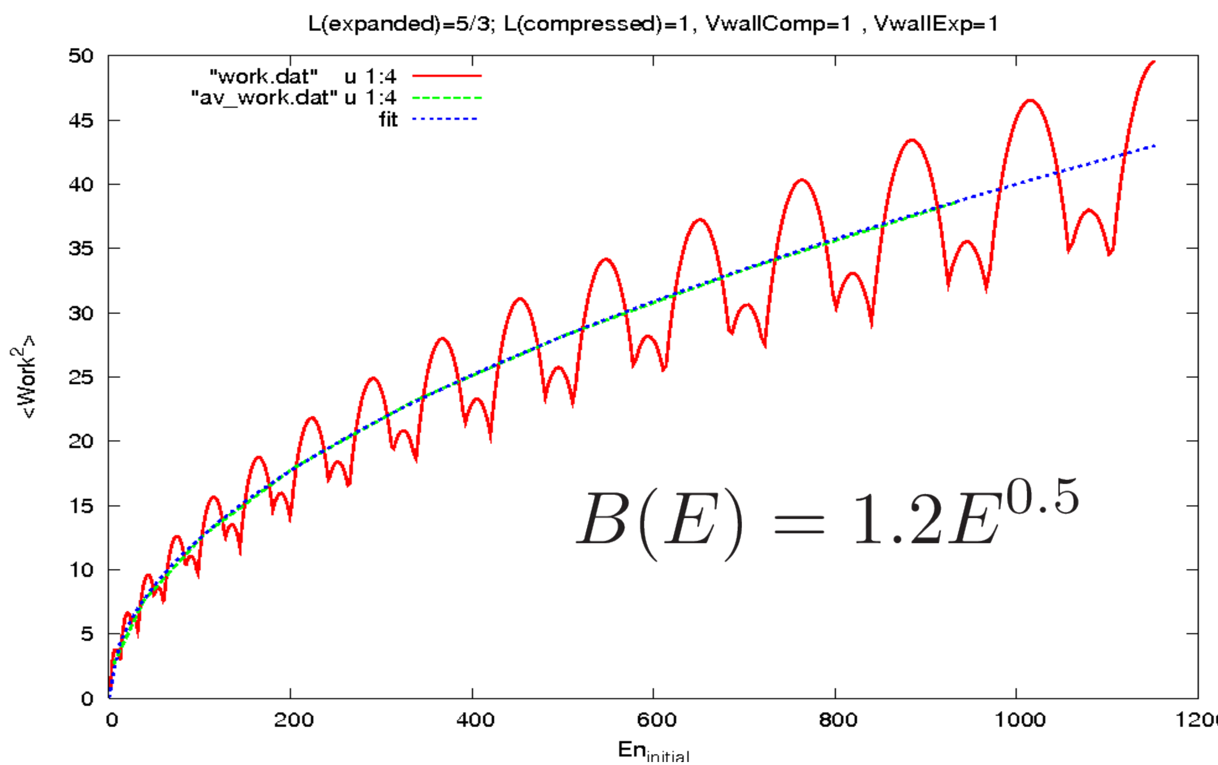
$$2A = \beta B + \partial_E B$$

This expression doesn't work

1D classical integrable (L=1, L'=5/3)



$$A(E) = 0$$



$$B(E) = 1.2E^{0.5}$$

$$2A = \beta B + \partial_E B$$

$$\frac{2-d}{2E}\sqrt{E} = \frac{1}{2}E^{-0.5}$$

Appendix 3: Jarzynski Equality (JE)

State initially in thermal equilibrium:

C. Jarzynski, Phys. Rev. E **56**, 5018–5035 (1997)

EXACT $$P(w)e^{-\beta W} = \widetilde{P}(-w) \rightarrow \langle e^{-\beta W} \rangle = 1$$

APPROXIMATE $$-\beta\langle W\rangle + \frac{\beta^2}{2}\langle \delta W^2 \rangle_c = 0 \rightarrow \boxed{2A = \beta B}$$

Any unitary evolution (there is no temperature here):

EXACT $$\Omega(E) T_{E\rightarrow E'} = \Omega(E')\widetilde{T}_{E'\rightarrow E}$$

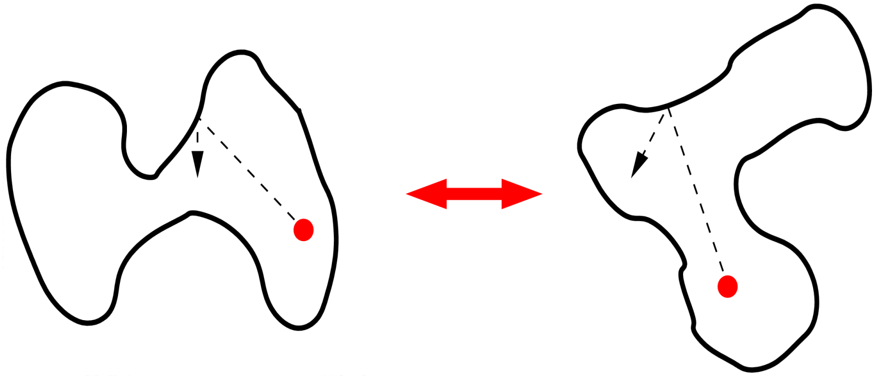
APPROXIMATE $$2A(E) = \beta(E)B(E) + \partial_E B(E)$$

"Generalized Einstein relation"

AVERAGE OVER DISTRIBUTION: $$\boxed{2A = \beta B + \left(1 - \frac{\sigma^2(E)}{\sigma^2_{eq}(E)}\right)\frac{\partial B}{\partial E}}$$

Example: particle in chaotic cavity

arXiv:1007.4589v2 & Physical Review E 83, 011107 (2011)



$$\left.\begin{aligned} A(E) &= gE^{1/2} \\ B(E) &= g\tfrac{4}{d+1}E^{3/2} \\ \beta(E) &= \tfrac{d-2}{2E} \end{aligned}\right\} \to 2A = \beta B + \partial_E B$$

 # d.o.f=2d

$$f(v,\tau)d\mathbf{v} \sim e^{-v/\tau}d\mathbf{v} \Rightarrow \frac{\sigma^2(E)}{\sigma^2_{eq}(E)} = \frac{2+3/d}{1+1/d} \to 2$$

$$\frac{\partial f}{\partial \tau} = d\frac{\partial f}{\partial v} + v\frac{\partial^2 f}{\partial v^2}$$

$$f(V,\tau) \sim e^{-\frac{V}{\tau}} \to f(E,t) \sim e^{-\sqrt{E}}$$