Nonequilibrium Dynamics of Bosons in Optical Lattices from Schwinger-Keldysh Calculation

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1. **Schwinger-Keldysh Formalism**
2. **Real-Time Ginzburg-Landau Theory**
3. **Collective Excitation Spectra**
4. **Collapse and Revival**

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Bose-Hubbard Model

Split Bose-Hubbard Hamiltonian:

$$\hat{H}_{BH} = \underbrace{\sum_i \left[\frac{U}{2}\hat{a}_i^\dagger\hat{a}_i\left(\hat{a}_i^\dagger\hat{a}_i - 1\right) - \mu\hat{a}_i^\dagger\hat{a}_i\right]}_{\equiv\hat{H}_0} \underbrace{- J\sum_{\langle i,j\rangle}\hat{a}_i^\dagger\hat{a}_j}_{\equiv\hat{H}_1}$$

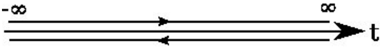
- Unperturbed system: Local interaction term is easily diagonalized by occupation number basis $\prod_i |n_i\rangle$
- Hopping as a perturbation physically justified?
  - Critical hopping for $D = 3$: $J/U \lesssim 1/30$
  - Dimensional scaling: $J \to J/D$ in presence of a condensate. For large D, higher-order hopping terms are suppressed

    K. Byczuk and D. Vollhardt, PRB **77**, 235106 (2008)
  - Resummed hopping expansion in first order is exact for $D = \infty$ ($1/D$ expansion)

Schwinger-Keldysh Formalism

- Consider contour-ordered Green functions:
  $\left\langle \hat{\mathrm{T}}_{\mathrm{c}} \left\{ \hat{\mathrm{U}}_{\mathrm{D}}(t, t_0)\hat{\mathrm{U}}^{\dagger}_{\mathrm{D}}(t, t_0)\hat{\mathrm{a}}^{\dagger}_{i\pm\mathrm{D}}(t_1)\cdots\hat{\mathrm{a}}_{j\pm\mathrm{D}}(t_n) \right\} \right\rangle_{\rho}$
- Remaining problem: Thermal averaging with respect to $\hat{\mathrm{H}}$!
- Solution: $\hat{\rho} = e^{-\beta\hat{\mathrm{H}}}/\mathcal{Z} = e^{-\beta\hat{\mathrm{H}}_0}\hat{\mathrm{U}}_{\mathrm{D}}(-i\hbar\beta, 0)/\mathcal{Z}$.
  - Interaction contour:

  - Keldysh countour: Push t_0 to infinite past!



Path-ordered Green functions

$$G^{\pm\cdots\pm}_{i\cdots j}(t_1\cdots t_n) \equiv$$
$$i^{n-1}\left\langle \hat{\mathrm{T}}_{\mathrm{c}} \left\{ \hat{\mathrm{U}}_{\mathrm{D}}(\infty, -\infty)\hat{\mathrm{U}}^{\dagger}_{\mathrm{D}}(\infty, -\infty)\hat{\mathrm{a}}^{\dagger}_{i\pm\mathrm{D}}(t_1)\cdots\hat{\mathrm{a}}_{j\pm\mathrm{D}}(t_n) \right\} \right\rangle_{0}$$

Generating Functional

Add source term $\hat{\mathrm{H}}_{\mathrm{Q}}$ to the Hamiltonian:

$$\hat{\mathrm{H}}_{\mathrm{Q}}[j, j^*](t) = \sum_i \left[j_i(t) \hat{\mathrm{a}}_i^\dagger(t) + \mathrm{h.c.} \right]$$

- Breaks the $U(1)$ symmetry of the BH-Hamiltonian!
- Allows to define generating functional:

$$\mathcal{Z}[j, j^*] \equiv \left\langle \hat{\mathrm{T}}_{\mathrm{c}} \exp \left\{ -\frac{i}{\hbar} \sum_{\pm} \pm \int_{-\infty}^{\infty} \mathrm{d}t \left[\hat{\mathrm{H}}_{1\pm}(t) + \hat{\mathrm{H}}_{\mathrm{Q}\pm}[j, j^*](t) \right] \right\} \right\rangle_0$$

- Expansion in j, j^*: Full Green's functions
- Expansion in some smallness parameter of $\hat{\mathrm{H}}_1$: Unperturbed Green's functions

Expansion of $\mathcal{F}[j, j^*] \equiv -i \ln \mathcal{Z}[j, j^*]$ contains only connected diagrams

Real-Time Ginzburg-Landau Theory

Conjugates of currents are order fields:

$$\hbar \delta \mathcal{F} / \delta j_\alpha^* = \langle \hat{\mathrm{a}}_\alpha \rangle \equiv \Psi_\alpha$$

Legendre transformation of $\mathcal{F}$ yields effective action:

$$\Gamma[\Psi, \Psi^*] \equiv \mathcal{F}[j, j^*] - \frac{1}{\hbar} \left(j_\alpha \Psi_\alpha^* + \mathrm{c.c.} \right)$$

Expansion in hopping (1st order) and fields (4th order):

$$\begin{aligned} \Gamma = \quad & \Psi_\alpha \left[C_{\alpha\beta}^{-1} - J_{\alpha\beta} \delta_{\mathrm{P}_\alpha, \mathrm{P}_\beta} \mathrm{P}_\alpha \right] \Psi_\beta^* \\ & - \frac{1}{4} C_{\alpha\beta\gamma\delta} C_{\alpha\alpha'}^{-1} \Psi_{\alpha'} C_{\beta\beta'}^{-1} \Psi_{\beta'} C_{\gamma'\gamma}^{-1} \Psi_{\gamma'}^* C_{\delta\delta'}^{-1} \Psi_{\delta'}^* \end{aligned}$$

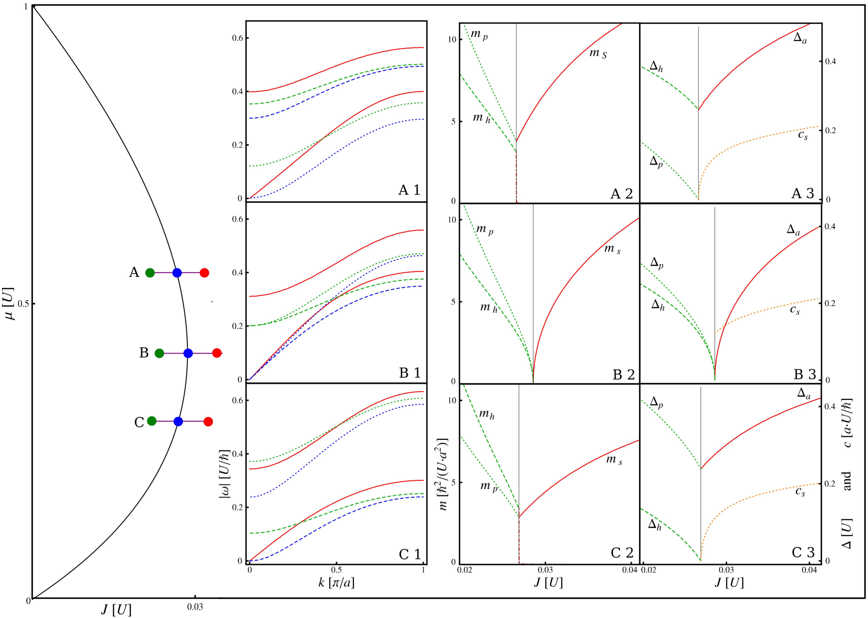
Comparison of Γ with imaginary-time formalism

[F.E.A. dos Santos and A. Pelster, PRA **79**, 013614 (2009)] and

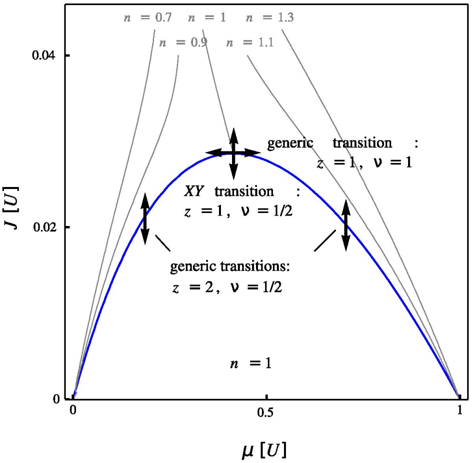
[B. Bradlyn, F.E.A. dos Santos and A. Pelster, PRA **79**, 013615 (2009)] yields

agreement for $T = 0$, but discrepancies for $T \neq 0$

Collective Excitation Spectra near Phase Boundary at $T = 0$



Critical Exponents



[M. P. A. Fisher *et al.*, PRB **40**, 546 (1989)]

- Scaling behavior:
  $\Delta \sim (J - J_{\mathrm{PB}})^{z\nu}$.
- Two universality classes:
  - Generic transition:
    driven by density variation
    $z\nu = 1$ (=mean field)
  - *XY*-like transition:
    driven by hopping variation
    $z\nu = 1/2$ (only at the tip!)

SF modes

- Gapped mode $\rightarrow$ lattice modulation
  [T. Stöferle *et al.*, PRL **92**, 130403 (2004)]
- Sound mode $\rightarrow$ Goldstone theorem, Bragg spectroscopy
  [P. T. Ernst *et al.*, Nat. Phys. **6**, 56 (2009)]
- Two modes $\rightarrow$ Phase/amplitude excitations
  [S. D. Huber *et al.*, PRB **75**, 085106 (2007)]
  [U. Bissbort *et al.*, PRL **106**, 205303 (2011)]

Deep in the SF phase:

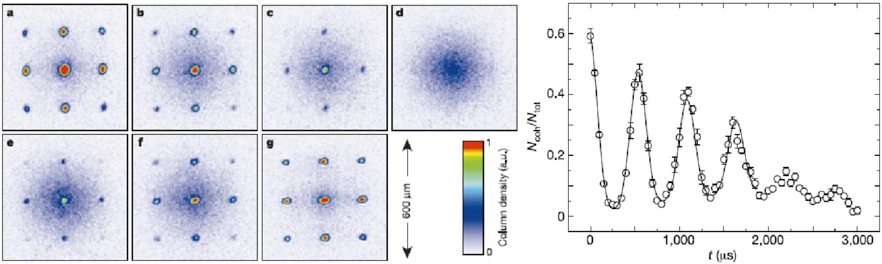
- Hopping expansion is not supposed to be good far away from phase boundary
- Nevertheless: Consider $U \ll J, \mu$ and expand in U
  $\Rightarrow$ Gross-Pitaevski equation: $i\hbar \frac{\partial \Psi_i}{\partial t} = -\sum_j J_{ij} \Psi_j - \mu \Psi_i - U \Psi_i |\Psi_i|^2$
- Yields Bogoliubov sound mode:
  $\hbar\omega(\boldsymbol{k}) = \sqrt{\left(4J\sum \sin^2(k_i a/2)\right)^2 + 2nU\left(4J\sum \sin^2(k_i a/2)\right)}$
- Gapped mode: Solve equation of motion first, and apply the limit $U \rightarrow 0$ then $\Rightarrow \omega = 2|\mu|$

Collapse and Revival of Matter Waves

Inhomogeneous Bose-Hubbard Hamiltonian:

$$\hat{H}_{\rm BH} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \sum_i \left[\frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu_i \hat{n}_i \right], \qquad \mu_i = \mu - \frac{m}{2} \omega^2 \mathbf{x}_i^2$$

Experiments:



- Periodic potential depth was suddenly changed from $V_A = 8\ E_r$ to $V_A = 22\ E_r$
- Collapse and revival was observed in a sample of $2 \times 10^5\ {}^{87}$Rb atoms: M. Greiner *et al.* Nature **419**, 51 (2002)

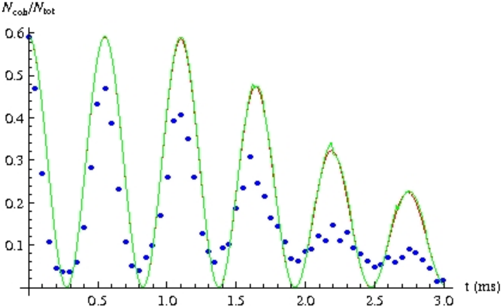
Results from Ginzburg-Landau Theory

- Measured coherent fraction:

$$N_{\text{coh}} = \int_{-\delta k}^{\delta k} dk_x \int_{-\delta k}^{\delta k} dk_y \int_{-\pi/a}^{\pi/a} dk_z |\psi(\mathbf{k}, t)|^2$$

- Approximation for large times:

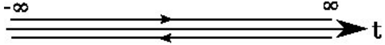
$$N_{\text{coh}} = \sum_{i=-[\frac{\delta k \hbar}{\omega^2 ta}]}^{[\frac{\delta k \hbar}{\omega^2 ta}]} \sum_{j=-[\frac{\delta k \hbar}{\omega^2 ta}]}^{[\frac{\delta k \hbar}{\omega^2 ta}]} \sum_{k=-\infty}^{\infty} |\psi_{ijk}(t)|^2$$



- blue dots: Experimetal data
- solid green: Numerical solution
- solid red: Large-time approximation

Summary and Outlook:

- Derivation from Bose-Hubbard Hamiltonian within non-perturbative hopping expansion
  - Imaginary-time Ginzburg-Landau theory
    [F.E.A. dos Santos and A. Pelster, PRA **79**, 013614 (2009)]
    [B. Bradlyn, F.E.A. dos Santos and A. Pelster, PRA **79**, 013615 (2009)]
  - Real-time Ginzburg-Landau theory
    [T.D. Grass, F.E.A. dos Santos, and A. Pelster, Phys. Rev. A **84**, 013613 (2011)]
- Agreement for $T = 0$
  - Collective excitation spectra
  - Collapse and revival dynamics
- Discrepancies for $T \neq 0$ due to Keldysh contour



- To do: Ginzburg-Landau theory with interaction contour