Quantum Fisher information as efficient entanglement witness in quantum many-body systems

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Entanglement: Resource of Quantum Simulation

Too much entanglement for classical computer

Trotzky et al., *Nat. Phys.* (2012)
Entanglement: Fingerprint of Quantum Phases

Example: Disordered systems and many-body localization

\[ S_{VN} \]

Many-body localized
Anderson localized

\[ \log(Jt) \]

Znidaric, Prosen, Prelovsek, *PRB* 2008
Bardarson, Pollmann, Moore, *PRL* 2012
Vosk and Altman, *PRL* 2013
Serbyn, Papic, Abanin, *PRL* 2013
The problem with quantifying many-body entanglement

Entanglement measures

$S_{\text{Rényi2}}$

$S_{\text{vN}}$

$\mathcal{E}_{\text{geom}}$

Problem 1:
Which is the relevant one?

Problem 2:
Non-linear functions of density matrix

$\mathcal{E} = f_{\text{nl}}(\rho)$

Exponential number of measurements!

Entanglement entropy
Theory: Daley, Pichler, Schachenmayer, Zoller, PRL 2013
Entanglement is really hard to measure in experiment.
Workaround: entanglement witnesses

Toth, Gühne, Cramer, Brukner, Lewenstein. . .
Workaround: entanglement witnesses
Toth, Gühne, Cramer, Brukner, Lewenstein.

The art is to find witnesses that
• are easy to measure
• but also tell us something relevant

Here:
Quantum Fisher Information
+ genuine multipartiteness
+ efficient at $T=0$ and $T>0$
+ interesting many-body settings
Content

Background Quantum Fisher Information
  metrological definition
  witness for multipartite entanglement

Measurability via response functions

QFI in quantum many-body systems
  Divergent entanglement at quantum phase transition
  Fingerprint of many-body localization
  Measure of coherence

Conclusions
Content

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Quantum Fisher Information $F_Q$ bounds parameter estimation

Braunstein, Caves *PRL* 1994

Quantum Cramer-Rao bound

$$(\Delta \theta)^2 \geq \frac{1}{F_Q \# \text{meas.}}$$

How sensitive is a state $\rho$ to change in parameter $\theta$?

= distinguishability of $\rho$ from

$$\rho' = e^{i \mathcal{O} \delta \theta} \rho e^{-i \mathcal{O} \delta \theta}$$

$$\sum_i \sigma_i^x h_x$$

squeezed state

entangled but not squeezed

See experiments groups
Oberthaler, Klempt, Bollinger

Quantum Cramer-Rao bound

$squeezed$ $state$

$entangled$ $but$ $not$ $squeezed$

e.g., Ma et al.,

*Phys. Rep.* 2011
It witnesses multipartite entanglement
Pezzé and Smerzi *PRL* 2009, Hyllus et al., *PRA* 2012, Toth *PRA* 2012

Precision limit of estimation

\[(\Delta \theta)^2 \geq \frac{1}{F_Q \# \text{meas.}}\]

Need entanglement for

\[F_Q > N\]

Need \(k + 1\)-body entanglement

\[F_Q/N > k\]
Calculation of the QFI
Braunstein, Caves *PRL* 1994

\[
F_Q = F_Q [\rho(\theta), \mathcal{O}]
\]

\[
\rho(\theta) \leftrightarrow \rho(\theta + \delta \theta) = e^{i\mathcal{O}\delta\theta} \rho(\theta) e^{-i\mathcal{O}\delta\theta}
\]

Pure states
\[
F_Q = 4 \left( \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2 \right)
\]

Mixed states
\[
\rho = \sum p_\lambda |\lambda\rangle \langle \lambda| \\
F_Q = 2 \sum_{\lambda, \lambda'} \frac{(p_\lambda - p_{\lambda'})^2}{p_\lambda + p_{\lambda'}} |\langle \lambda | \mathcal{O} | \lambda' \rangle|^2
\]

How to measure this?
Content

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Conclusions
Content

Background Quantum Fisher Information
  metrological definition
  witness for multipartite entanglement

Measurability via response functions

QFI in quantum many-body systems
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How sensitive is a state $\rho$ to change in parameter $\theta$?

$= \text{distinguishability of } \rho \text{ from } \rho' = e^{i\hat{O}\delta\theta} \rho e^{-i\hat{O}\delta\theta}$

$(\Delta \theta)^2 \geq \frac{1}{F_Q \# \text{meas.}}$

See experiments groups
Oberthaler, Klempt, Bollinger

squeezed state
entangled but not squeezed

entanglement
e.g., Ma et al.,
*Phys. Rep.* 2011
In condensed matter: Sensitivity of state = susceptibility

Static susceptibility: 
\[ H = H_0 + h \mathcal{O} \]
\[ M = \langle \mathcal{O} \rangle \]
\[ \chi = \frac{\partial M}{\partial h} \]

Dynamic susceptibility: 
\[ H = H_0 + h \cos(\omega t) \mathcal{O} \]
\[ \chi(\omega, T) = i \int_0^\infty dt \, e^{i\omega t} \langle [\mathcal{O}(t), \mathcal{O}] \rangle_T \]
\[ \chi''(\omega, T) = \Im(\chi(\omega, T)) \]

Response function routinely measured in neutron scattering, Bragg spectroscopy, ...

Sengstock group
*Nat. Phys.* 2010
Quantum Fisher Information and dynamic susceptibility are the same (in thermal states)

\[ F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \tanh \left( \frac{\omega}{2T} \right) \chi''(\omega, T) \]


+ Makes QFI directly measurable

+ Connects notions of sensitivity from two fields

+ Allows efficient calculations and derivation of scaling laws

See also
Kolodrubetz, Mehta, Polkovnikov, arXiv:1602.01062
Greschner, Kolezhuk, and Vekua, PRB 2013
QFI = susceptibility  –  a short proof

Have
\[ \chi(\omega, T) = i \int_0^\infty dt \, e^{i\omega t} \langle [\mathcal{O}(t),\mathcal{O}] \rangle_T \quad \text{and} \quad \chi''(\omega, T) = \Im(\chi(\omega, T)) \]

Lehmann representation (energy eigenbasis)  
\[ \rho = \sum_{\lambda,\lambda'} (p_\lambda - p_{\lambda'}) |\lambda\rangle \langle \lambda|, \quad p_\lambda = e^{-\beta E_\lambda} / Z \]
\[ \chi''(\omega) = \sum_{\lambda,\lambda'} (p_\lambda - p_{\lambda'}) |\langle \lambda|\mathcal{O}|\lambda' \rangle|^2 \pi \delta(\omega + E_{\lambda'} - E_\lambda) \]

Use  
\[ 2 \int_0^\infty d\omega \, \tanh \left( \frac{\omega}{2T} \right) \delta(\omega + E_{\lambda'} - E_\lambda) = \tanh \left( \frac{E_{\lambda'} - E_\lambda}{2T} \right) = \frac{p_\lambda - p_{\lambda'}}{p_\lambda + p_{\lambda'}} \]

\[ F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \, \tanh \left( \frac{\omega}{2T} \right) \chi''(\omega, T) \]

Independent of microscopic details  
( only thermal states,  
Kubo linear response )

Want  
\[ F_Q = 2 \sum_{\lambda,\lambda'} \frac{(p_\lambda - p_{\lambda'})^2}{p_\lambda + p_{\lambda'}} |\langle \lambda|\mathcal{O}|\lambda' \rangle|^2 \]
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Measurability via response functions

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Conclusions
Content

Background Quantum Fisher Information
  metrological definition
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Example

Ising chain in transverse field

\[ H = -J \sum \sigma_i^x \sigma_{i+1}^x + h \sum \sigma_i^z \]

ferromagnet

quantum phase transition

paramagnet
Divergence of many-body entanglement at critical point

For $N=64$ at least 13-body entanglement

Entangled region at $T>0$
see also Tóth, and Wu et al., *PRA* 2005

Entanglement peak at criticality

Test scaling
Strong scaling in Ising chain

Finite-size
\[ f_Q \approx L^{3/4} \phi_{Q_1}(LT) \]

Finite-temperature
\[ f_Q \approx T^{-3/4} \phi_{Q_2}(LT) \]

\[ f_Q = c T^{-3/4} \] from known scaling of \( \chi''(\omega, T) \)


\[ F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \tanh \left( \frac{\omega}{2T} \right) \chi''(\omega, T) \]


see also works by Zanardi, Campos-Venuti, Gu, and others for metric tensor/fidelity susceptibility
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Conclusions
Content

Background Quantum Fisher Information
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QFI as fingerprint for exotic quantum behavior: many-body localization (MBL)


Entanglement growth and MBL, e.g.:
Znidaric, Prosen, Prelovsek, *PRB* 2008
Bardarson, Pollmann, Moore, *PRL* 2012
Vosk and Altman, *PRL* 2013
Serbyn, Papic, Abanin, *PRL* 2013
Indications for log-growth in trapped-ion experiment

Variance
[= QFI if state pure]

(a) No Disorder

(b) $W = 6J_{\text{max}}$

$W = 8J_{\text{max}}$
Content

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  Fingerprint of many-body localization
  Measure of coherence

Conclusions
Content

Background Quantum Fisher Information
  metrological definition
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Measurability via response functions

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Resource theoretic definitions of coherence

1. Define what is an incoherent state
2. Define what are “free” operations (do not generate coherence)
3. Define measure for coherence: does not increase under free operations
Resource theoretic definitions of coherence

Baumgratz, Cramer, Plenio, *PRL* 2014

Coherence with respect to a basis
e.g., $S_i^z$  

$\rho_{\text{inc.}} = \begin{pmatrix}
|\uparrow\uparrow\uparrow\rangle |\uparrow\uparrow\downarrow\rangle \\
|\uparrow\downarrow\uparrow\rangle |\downarrow\downarrow\rangle
\end{pmatrix}$

Marvian and Spekkens, several works

Coherence with respect to a Hamiltonian
e.g., $H = \sum_i S_i^z$

Resource for various tasks:
reference-frame alignment,
thermodynamic tasks, quantum metrology

$F_Q$ is a strict measure for coherence
Coherence in NMR: multi-quantum coherences

Collaboration M. Gärtnner, A. Safavi-Naini, M. Wall, and A. M. Rey

Different magnetization sectors

\[ S_z |M_z\rangle = M_z |M_z\rangle \]

\[ \rho = \sum_{m=-N}^{N} \rho^{(m)} \quad \rho^{(m)} = \sum_{M_z} \rho_{M_z M_z - m} |M_z\rangle \langle M_z - m| \]

multi-quantum coherences

\[ I_m = \text{tr}(\rho^{(-m)} \rho^{(m)}) \]

Multi-quantum coherence and 
quantum Fisher information

Martin Gaerttner et al., in preparation

\[ 2 \sum_{m=-N}^{N} I_m m^2 \leq F_Q [\rho, S_z] \leq 4d \sqrt{\sum I_m m^2} \]

\( F_Q \) is a strict measure for coherence (in the Marvian-Spekkens sense)
\( I_m \) are actually only witnesses for coherence
\( I_m \) are also witnesses for entanglement
Content

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Conclusions
Content

Background Quantum Fisher Information
  metrological definition
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• Connected Quantum Fisher Info and dynamic susceptibility
  – measure QFI efficiently, also at T>0
  – independent of microscopic details
  – Scaling theory for QFI (predictions from known critical exponents)

\[ F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \tanh \left( \frac{\omega}{2T} \right) \chi''(\omega, T) \]

• QFI: from metrology to quantum many-body systems
  – divergent entanglement at quantum phase transition
  – fingerprint of many-body localization
  – measure of coherence as resource

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