Running Coupling (after P. Nason, CERN 98-03, ESHEP '97)

- Physical quantity $G = G(\alpha, Q^2, \text{kinematics} \ldots)$
- if theory is renormalizable: write renormalised coupling α^r as series in α: $\alpha^r = \alpha\,(1 + c_1\alpha + c_2\alpha^2 + \ldots)$

  $\to$ α dimensionless; coefficients $c_i = c_i\left(\frac{Q^2}{\mu^2}\right)$, cannot depend on any scale
- G could also be expressed as $G(\alpha^r, \mu^2, \text{kinematics})$
- RGE: change (α^r, μ^2) but keep (α, Q^2) fixed $\to$ <u>same</u> physics

$$\frac{\partial G(\alpha^r,\mu^2)}{\partial \alpha^r}\, d\alpha^r + \frac{\partial G(\alpha^r,\mu^2)}{\partial \mu^2}\, d\mu^2 \overset{!}{=} 0$$

$$\frac{\partial \alpha\left(\alpha^r, \frac{Q^2}{\mu^2}\right)}{\partial \alpha^r}\, d\alpha^r + \frac{\partial \alpha\left(\alpha^r, \frac{Q^2}{\mu^2}\right)}{\partial \mu^2}\, d\mu^2 \overset{!}{=} 0$$

$$\Rightarrow \boxed{\mu^2 \frac{d\alpha^r}{d\mu^2} = -\mu^2 \frac{\frac{\partial \alpha\left(\alpha^r, \frac{Q^2}{\mu^2}\right)}{\partial \mu^2}}{\frac{\partial \alpha\left(\alpha^r, \frac{Q^2}{\mu^2}\right)}{\partial \alpha^r}} = -\frac{\mu^2 \frac{\partial G(\alpha^r,\mu^2)}{\partial \mu^2}}{\frac{\partial G(\alpha^r,\mu^2)}{\partial \alpha^r}} \equiv \beta(\alpha^r)}$$

$$\underbrace{\text{no } \mu^2 \text{ dependence } \left(\text{only } \tfrac{Q^2}{\mu^2}\right) \qquad \text{no } Q^2 \text{ dependence}}_{\beta \text{ can only depend on } \alpha^r}$$

- β function: $\alpha\left(\alpha^r, \frac{Q^2}{\mu^2}\right) = \alpha^r\left(1 + d_0\left(\frac{Q^2}{\mu^2}\right)\alpha^r + d_1\left(\frac{Q^2}{\mu^2}\right)(\alpha^r)^2 + \ldots\right)$

$$\rightsquigarrow \beta = \mu^2 \frac{d\alpha^r}{d\mu^2} = \mu^2 (\alpha^r)^2 \cdot \frac{\partial d_0}{\partial \mu^2} + \ldots \quad \text{independent of } Q^2, \mu^2!$$

$$\Rightarrow \mu^2 \frac{\partial d_i}{\partial \mu^2} \overset{!}{=} \text{const} \equiv -\beta_i \;, \text{ e.g. } d_0 = \beta_0 \ln\frac{Q^2}{\mu^2}$$

- 1 Loop Solution: $\alpha^s(Q^2) = \dfrac{\alpha^s(\mu^2)}{1+\beta_0\,\alpha^s(\mu^2)\ln\frac{Q^2}{\mu^2}}$

- $\alpha^s(\mu^2)$ arbitrary & dim. less $\rightarrow$ convert to "typical" reference scale

$\rightarrow$ QCD: scale at which theory gets non-perturbative

$\rightarrow$ $1 \overset{!}{=} \beta_0\,\alpha^s(\mu^2)\ln\frac{\mu^2}{\Lambda^2}$

$\Rightarrow$ $\alpha^s(Q^2) = \dfrac{1}{\beta_0 \ln\frac{Q^2}{\Lambda^2}}$ (correct running only for $Q^2 > \Lambda^2$!)