The Foucault Pendulum's Trajectory - the Formalism

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A few words before we start:

A Foucault pendulum is, at first sight, a rather simple device: Quite a normal pendulum, driven by gravity and inertia, whose plane of oscillation moves clockwise - at least in the northern hemisphere of our planet - due to the Earth's proper rotation about its own axis.

But at a closer look, there are a couple of sources of perturbation and distortion which we will shortly mention here but rather neglect when deriving the formula for the pendulum's trajectory.

First, when the pendulum swings towards the position of maximum amplitude, it will be lifted a bit so that it gains some potential energy - else there would be no retracting force at all. The pendulum moves in three - dimensional space. But we will treat it as moving only in a plane, which is justified by the smallness of the vertical displacement.

Second, the pendulum rope, 11 m long in the KIP case, can easily be excited to small transverse oscillations. These, together with tiny perturbations due to mechanical imperfections, can cause the pendulum to deviate fom planar swinging and start a rather elliptical path such that the ellipse performs sort of "Lissajous motion" which is not caused by the Coriolis force, but is rather due to phase differences between the movements of small and big half - axis.

Finally, our device is treated as a "mathematical pendulum", i.e. a point mass hanging on a massless, infinitely thin rope. This is again (rather) justified by the dimensions of pendulum bob and rope.

In short: The formula derived below is somewhat idealised as the above mentioned effects are not taken into account. But these are small, and the pendulum trajectory observed is rather well described even by a simplified formula.

The formalism:

There are two forces acting upon the pendulum: Gravity and Coriolis force. Suppose we have a pendulum hanging from a rope of length l, in its equilibrium position. Then, moving it out of this position by an angle ϕ, gravity will try to move it back with a component of $-m\ g\ \sin\phi$. Here m is the mass of the pendulum bob, and g is the gravitational acceleration at the point where the pendulum is situated. If the displacement is amall when compared to the length of the rope, we are allowed to use the "small angle approximation", i.e. $\sin\phi\ \approx\ \phi$, and the retracting force will be $-m\ g\ \phi$. From this we get immediately the equation of motion due to gravitation:

$$m\ (l\ \ddot{\phi})\ +\ m\ g\ \phi\ =\ 0.$$

(We note here that, in the small angle approximation, we may set $l\ \ddot{\phi}\ =\ \ddot{x}$ if, for example, we let the pendulum swing along an x - axis. We will adopt this notation later.) Remember: $\ddot{\phi}\ =\ \partial^2\phi/\partial\ t^2$.

We know from our high school days that the solution of this differential equation is a harmonic oscillation with frequency/angular velocity

$$\omega_P\ =\ 2\ \pi/T_P\ =\ \sqrt{g/l}.$$

The Coriolis force, in vector notation, reads

$$\vec{f_C}\ =\ 2\ m\ \vec{v}\ \times\ \vec{\omega}_E\ \ .$$

Here, $\vec{v}$ is the velocity of the pendulum bob (after our simplification in two dimensions only!), and $\vec{\omega}_E$ is the vector of the Earth's angular velocity whose modulus is $2\ \pi/24\ h$, corresponding to $360°/24\ h$ when expressed in degrees.

However, the effective angular velocity of our planet which enters the formula for the Coriolis force gets smaller when we move from one of the poles towards the equator: The "vertical" component at a place with latitude θ is only $\omega_E\ \sin\theta$ so that we have to write our formula for an arbitrary latitude as

$$\vec{f_C}\ =\ 2\ m\ \vec{v}\ \times\ \vec{\omega}_{E,\theta}$$

with the modulus

$$f_C\ =\ 2\ m\ v\ \omega_E\ \sin\theta\ \ .$$

The angle between $\vec{v}$ and $\vec{\omega}_E$ is always 90° as long as $\vec{v}$ is constraint to the Earth's surface. Now we put things together and sort according to components x and y. Our frame of reference is our laboritory where we observe the pendulum's enigmatic movements - fixed upon the Earth's surface -, and we are free to choose the x - axis e.g. parallel to a meridian, pointing to the north pole, and the y - axis pointing east along a circle of latitude. Then, with ω_P as the pendulum's circular frequency, we may write the components of our forces as follows:

$$f_{g,x}\ =\ -\ m\ \omega_P^2\ x,$$
$$f_{g,y}\ =\ -\ m\ \omega_P^2\ y,$$

$$f_{C,x}\ =\ 2\ m\ \omega_E\ \dot{y}\ \sin\theta,$$
$$f_{C,y}\ =\ -\ 2\ m\ \omega_E\ \dot{x}\ \sin\theta.$$

The first two equations apply to gravitation, the second two apply to the Coriolis force. The factor $\sin\theta$ accounts for the angle of latitude which is zero at the equator and 90° at the north pole. Now we get immediately two differential equations for the x- and y-components of the pendulum bob's trajectory:

$$\ddot{x} = -\omega_P^2\, x + 2\,\omega_E\, \dot{y}\, \sin\theta,$$
$$\ddot{y} = -\omega_P^2\, y - 2\,\omega_E\, \dot{x}\, \sin\theta.$$

Now, to make life easier, we replace the real plane on Earth's surface with the complex plane, i.e. we set $z = x + i\, y$ and get an elegant expression:

$$\ddot{z} + 2\, i\, \omega_E\, \dot{z}\, \sin\theta + \omega_P^2\, z = 0.$$

Experience tells us that a differential equation containing a function and its first and second derivative is generally solved with the help of an exponential. We start therefore with an (educated!) guess and make the ansatz

$$z(t) = Z_0(t) \cdot exp(-i\omega_E\, \sin\theta\; t),$$

or for short, with $\alpha = \omega_E\, \sin\theta$,

$$z(t) = Z_0(t) \cdot exp(-i\alpha\; t).$$

Inserting this into the differential equation above we get

$$\ddot{Z(t)} + (\alpha^2 + \omega_P^2)\, Z(t) = 0.$$

The parameter α contains the angular velocity of the Earth's eigen - rotation, i.e. α^2 is tiny when compared to ω_P^2, and we may safely neglect it. This leaves us with

$$\ddot{Z(t)} + \omega_P^2\, Z(t) = 0,$$

which we solve throug the general expression

$$Z(t) = A\, exp(i\omega_P t) + B\, exp(-i\omega_P t).$$

Then the complete solution for our problem reads

$$z = exp(-i\alpha t) \cdot (A\, exp(i\omega_P t) + B\, exp(-i\omega_P t)).$$

We do immediately see that there are two special solutions which correspond to harmonic oscillations of our pendulum, namely the solutions with $A = B$, and $A = -B$ where A and B are real. In the first case, we get by virtue of Euler's formulae, the form

$$z = 2\, A\, exp(-i\alpha t) \cdot \cos\omega_P t,$$

and in the second case we get

$$z = 2\, i\, exp(-i\alpha t) \cdot \sin\omega_P t.$$

Apparently, the first solution corresponds to the initial condition $z(t = 0) = 2\, A$, and the second one to $z(t = 0) = 0$. Without the factor $exp(-i\; \alpha\; t)$ which is due to the Coriolis force we would simply get the equations for a normal pendulum, swinging in a plane.

Applying Euler's formulas once more, we get rid of the exponential and obtain

$$z = 2\, A\, (\cos\alpha t - i \sin\alpha t \cdot \cos\omega_P t)$$

and

$$z \;=\; 2\,iA\,(\cos\alpha t \;-\; i\sin\alpha t \;\cdot\; \sin\omega_P t).$$

We note here that this corresponds to the stationary solution, i.e. valid for t from $-\infty$ to $+\infty$.

Of course, we would like to get an impression of our solution as a visible trajectory. To do this, we have to disentangle the complex equations again into real and imaginary parts. This gives for the first case, the "cosine case",

$$Re(z) \;=\; x \;=\; 2\;A\;\cos\alpha t\cdot\cos\omega_P t,$$
$$Im(z) \;=\; y \;=\; -\,2\;A\;\sin\alpha t\cdot\cos\omega_P t,$$

and for the second case, the "sine case",

$$Re(z) \;=\; x \;=\; 2\;A\;\sin\alpha t\cdot\sin\omega_P t,$$
$$Im(z) \;=\; y \;=\; 2\;A\;\cos\alpha t\cdot\sin\omega_P t.$$

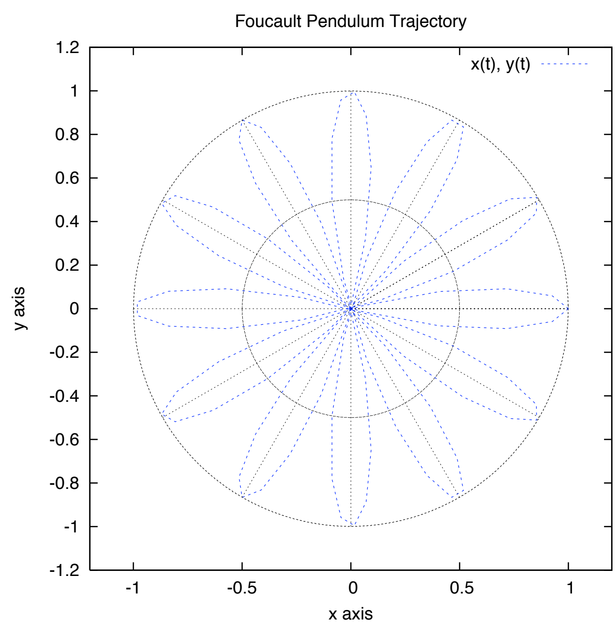

Figure 1: Simplified Foucault Pendulum trajectory

Now we have the coordinates we need to visualize our pendulum's trajectory by a graph. In real life, the time our planet needs for a full revolution is 24 hours while the pendulum period is 6.7 seconds. We will therefore show a "blow up" where a "day" has only six pendulum periods. This is shown in the above figure.

The figure is based on the "cosine - formula". When plotting the "sine - formula", we get the same picture with the only difference that the rosetta - shaped trajectory is rotated by an angle corresponding to one fourth of the pendulum period - this is simply due to the fact the "sine - formula" is derived for another starting condition.

A Short Summary

Apparently, we are finished - let's have a short summary. The trajectory of our pendulum is, in any case, "rosetta shaped". Drawing the genuine trajectory of the KIP pendulum is rather impossible as it swings 12896 times during one day. The "rosetta leaves" will therefore be a bit narrow. Anyhow, there ARE rosetta leaves as shown by a simple gedanken experiment: Suppose, the pendulum rope (11 m in reality) would be elongated such that the situation drawn in our figure is reproduced: One day lasts in fact six pendulum periods. What we then would see is just a blow up of the genuine trajectory.

It is perhaps worth while to touch upon a possible misunderstanding. We have talked all the time of the Coriolis force which makes the pendulum plane rotate. But it is not this force which causes the strong and strongest curvatures of the trajectory. These occur when the radial velocity goes to zero while the azimuthal velocity (due to the Earth's rotation) remains constant. Or, to make use of another gedanken experiment: Let a normal pendulum swing in a fixed plane, and let some sugar drop out of the pendulum bob. Put a long strip of paper beneath the pendulum and draw it with constant velocity perpendicular to the pendulum swing. Then the sugar will draw a sine (or cosine) line upon the paper with more or less narrow bends, depending on how fast you move the paper. And there is no Coriolis force at all. A corresponding effect exerts wide or narrow bends on the Foucault pendulum trajectory.

The formalism presented here is, as said in the very beginning, for an idealised case with no perturbations. If you build your own pendulum, you will have the chance to observe a lot of perturbations, and to think about them. For those who want to dive deeper into the sea of inertia, we quote a couple of links which may lead you furher and further.. But, after all, our planet rotates about its axis, and will continue so forever. No!! - that is not true: The rotation frequency decreases slowly but steadily. But this is another story, and to understand it, look at the full moon in a clear summer night!

http://en.wikipedia.org/wiki/Foucault_pendulum
http://www.phys.unsw.edu.au/~jw/pendulumdetails.html
http://www.millersville.edu/~physics/exp.of.the.month/35/ELIPREC/corelips.htm
http://www.cleonis.nl/physics/phys256/coupling.php
http://geosci.uchicago.edu/~nnn/LAB/DEMOS/coriolis.html