Multiple atomic dark solitons in cigar-shaped Bose-Einstein condensates

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We consider the stability and dynamics of multiple dark solitons in cigar-shaped Bose-Einstein condensates. Our study is motivated by the fact that multiple matter-wave dark solitons may naturally form in such settings as per our recent work [Phys. Rev. Lett. 101, 130401 (2008)]. First, we study the dark soliton interactions and show that the dynamics of well-separated solitons (i.e., ones that undergo a collision with relatively low velocities) can be analyzed by means of particle-like equations of motion. The latter take into regard the repulsion between solitons (via an effective repulsive potential) and the confinement and dimensionality of the system (via an effective parabolic trap for each soliton). Next, based on the fact that stationary, well-separated dark multisoliton states emerge as a nonlinear continuation of the appropriate excited eigenstates of the quantum harmonic oscillator, we use a Bogoliubov-de Gennes analysis to systematically study the stability of such structures. We find that for a sufficiently large number of atoms, multiple soliton states are dynamically stable, while for a small number of atoms, we predict a dynamical instability emerging from resonance effects between the eigenfrequencies of the soliton modes and the intrinsic excitation frequencies of the condensate. Finally, we present experimental realizations of multisoliton states including a three-soliton state consisting of two solitons oscillating around a stationary one and compare the relevant results to the predictions of the theoretical mean-field model.

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1I. INTRODUCTION

Dark solitons, namely localized density dips on top of a stable continuous-wave background (or a background of finite extent) with a phase jump across their density minimum, are fundamental envelope excitations supported in nonlinear dispersive media. These nonlinear waves emerge in media with a positive (negative) group-velocity dispersion and defocusing (focusing) nonlinearity, with a proper model describing their evolution being the so-called defocusing nonlinear Schrödinger (NLS) equation [1]. Dark solitons have been studied extensively in the field of nonlinear optics (from which the term “dark” was coined) [2], but they have also been observed in diverse physical contexts, including liquids [3], mechanical systems [4], magnetic films [5], and so on.

More recently, dark solitons have attracted much attention in the physics of Bose-Einstein condensates (BECs) [6,7], where they appear as fundamental macroscopic nonlinear excitations of BECs with repulsive interatomic interactions [8,9]. It is, therefore, not surprising that experimental work on matter-wave dark solitons started as early as 10 years ago [10–14] and, very recently, continued even more intensively [15–19]. These important “new age” experiments highlighted various salient features of dark solitons, verified previous theoretical predictions and offered motivation for further investigations. A pertinent example is the creation of more than one matter-wave dark solitons [17,18] (see also Ref. [15]) in cigar-shaped condensates, which were allowed to interact. This invites a revisiting of the topic of dark solitons and especially of their interactions in the particular context of BECs; the latter, has a number of particularities including, e.g., the confinement that is routinely used to trap and cool the atomic cloud [6,7].

Multiple dark-soliton solutions of the defocusing NLS were first obtained in Ref. [1] by means of the inverse scattering method. Later, an analytical form of a solution of the NLS equation composed of two dark solitons of different depths and velocities was found [20] (see also the more recent works [21,22]), and it was shown that the interaction between dark solitons is repulsive. Subsequent theoretical studies focused on the interactions and collisions of dark solitons in the context of nonlinear optics [23–25] and later in BECs [26], while relevant experimental results (see Ref. [27] for optical dark solitons and [14,17] for atomic dark solitons) also examined the interaction between two dark solitons. However, following the recent experimental methodology of Ref. [18], it is in principle possible to generate multiple (in fact, in principle, an arbitrary number of) dark solitons: this can be done by releasing a BEC from a double-well potential into a harmonic trap within the experimentally accessible, “dimensionality” crossover regime between one dimension (1D) and three dimensions (3D) [18]. In such a case, it is clear that a study of multiple matter-wave dark solitons, and their interactions, should be performed in a theoretical framework that takes into regard basic features of the pertinent experiment, such as the effect of dimensionality and the corresponding effective confinement of the condensate.

In this work, our scope is to analyze this problem, namely the statics and dynamics of multiple matter-wave dark solitons in cigar-shaped condensates. Based on the fact that recent atomic dark-soliton experiments were performed at extremely low temperatures and with sufficiently large number of atoms, we may safely adopt a mean-field theoretical approach. In particular, we will perform our analysis in the framework of the effectively 1D Gross-Pitaevskii (GP) equation with a noncubic nonlinearity that was first presented in Ref. [28] and later was also derived and tested in Refs. [29] (this is a distinguishing feature of our work with respect to most of the above references which considered dark-soliton interactions in the standard homogeneous defocusing cubic NLS setting). Our analysis starts by considering the dynamics of multiple dark
solitons which is studied as follows. First, we consider the weakly interacting limit of the noncubic GP equation (namely the traditional defocusing NLS model) and, in the absence of the trap, we derive an effective repulsive potential for the interaction between two solitons. It is shown that this potential can successfully be used to describe the interactions between "low-speed" solitons (with velocities less than the half of the speed of sound). Such solitons are "well separated," in the sense that they can always be identified as distinguishable objects, even at the collision point. Then, using this potential, we obtain a set of equations of motion for the coordinates of an arbitrary number of solitons. Our approach is finally applied to the full problem under consideration (with the noncubic nonlinearity and the external harmonic trap), by incorporating an effective harmonic potential with a corresponding characteristic frequency. This frequency is the oscillation frequency of dark solitons, which, in the 1D Thomas-Fermi limit, takes the characteristic value \( \omega_z/\sqrt{2} \) (where \( \omega_z \) is the longitudinal trap frequency) [30]. This value is equal to the eigenfrequency of the first anomalous mode of the system [7], corresponding to the oscillation frequency of a single dark soliton in the trap (see relevant results in Refs. [31] and [32] for the purely 1D and dimensionality-crossover regimes). Such an ad hoc decomposition of the principal physical mechanisms affecting the solitons was introduced in Ref. [18] (for the case of two symmetrically interacting dark solitons) and will be validated a posteriori herein by means of direct numerical simulations.

The above methodology for the study of multiple atomic dark solitons is directly connected to the Bogoliubov-de Gennes (BdG) spectrum of excitations of stationary dark-soliton states. The latter is obtained when linearizing around the nonlinear counterparts of the respective linear states (corresponding to the eigenmodes of the quantum harmonic oscillator) [33] and their properties are studied by means of the well-known BdG equations [7]. Such an analysis reveals that the spectrum of the \( n \)-th excited state consists of one zero eigenvalue, \( n \) double eigenvalues (accounted for by the presence of the harmonic trap), and infinitely many simple ones. In the nonlinear regime, one of the eigenvalues of each double pair possesses a topological property of so-called negative energy (in the physical literature) [34] or negative Krein signature (in the mathematical literature) [35]; practically, this means that it becomes structurally unstable, i.e., it becomes complex on collision with eigenvalues of positive energy. The eigenvalues with negative Krein signature are actually associated with the anomalous modes [7] appearing in the BdG spectrum. In our case of multiple dark solitons, the number of anomalous modes in the excitation spectrum equals the number of dark solitons [36], which is in agreement with the fact that the number of eigenvalues with negative Krein signature equals to the number of the nodes of the stationary state [37]. More generally, we conjecture (based on the results below for the cases of two- and three-solitons) that in the case of an \( n \)-dark-soliton sequence (pertinent to an \( n \)-th order nonlinear state), the anomalous modes of the system correspond to the excitation of the normal modes of the "dark-soliton lattice."

The article is organized as follows. In Sec. II we present the model and make some general remarks on the theoretical setup. Section III is devoted to the dynamics of multiple solitons. In particular, first we analyze the homogeneous weakly interacting case and derive the effective repulsive potential for two solitons undergoing a symmetric collision. Then, we generalize these results to include the cases of asymmetric collisions and multiple solitons, as well as to tackle the full problem, taking into regard the external harmonic trap and the dimensionality of the condensate. In Sec. IV we study the stability of the stationary multisoliton states via a BdG analysis. We analyze, in particular, the pertinent Bogoliubov spectra, paying special attention to the anomalous modes of the system. We illustrate how these anomalous modes correspond to "normal modes" of the "dark-soliton-lattice," e.g., in-phase and out-of-phase oscillating dark soliton states. We also predict the onset of dynamical instabilities due to resonance between the eigenfrequencies of these normal modes and the excitation frequencies of the background condensate. We illustrate under what conditions such instabilities may be observed in future experiments. In Sec. V we present experimental realizations of multisoliton states, including a three-soliton state consisting of two solitons oscillating around a stationary one. Section VI concludes the article, summarizing our findings and presenting some directions of future study.

II. THE MODEL AND THEORETICAL SETUP

We consider a BEC confined in a highly elongated trap, with longitudinal and transverse confining frequencies (denoted by \( \omega_z \) and \( \omega_\perp \), respectively) such that \( \omega_z \ll \omega_\perp \). In this case, it can be found [28,29] that use of the adiabatic approximation, in combination with a variational approach for determining the local transverse chemical potential, leads to the following effective 1D GP equation,

\[
\frac{i\hbar}{\partial t} \psi = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_{\text{ext}}(z) + \hbar \omega_\perp \sqrt{1 + 4a|\psi|^2} \right] \psi, \tag{1}
\]

where \( \psi(z,t) \) is the longitudinal part of the condensate’s wavefunction normalized to the number of atoms, i.e., \( N = \int_{-\infty}^{\infty} |\psi|^2 dx \), \( a \) is the s-wave scattering length, \( m \) is the atomic mass, and \( V_{\text{ext}}(z) \) is the longitudinal part of the external trapping potential, assumed to take the standard harmonic form \( V_{\text{ext}}(z) = (1/2)\mu \omega_z^2 z^2 \). As demonstrated in Refs. [29], Eq. (1) provides accurate results in the dimensionality crossover and in the Thomas-Fermi limit, thus describing the axial dynamics of cigar-shaped BECs in a very good approximation to the 3D GP equation. Note that in the weakly interacting limit, \( 4a|\psi|^2 \ll 1 \), Eq. (1) is reduced to the usual 1D GP equation with a cubic nonlinearity, characterized by an effective 1D coupling constant \( g_{1D} = 2a\hbar \omega_\perp \). Equation (1) can be expressed in the following dimensionless form,

\[
\frac{1}{\Omega} \frac{d\psi}{dt} = \left[ -\frac{1}{2} \frac{\partial^2}{\partial \zeta^2} + \frac{1}{2} \Omega^2 \zeta^2 + \sqrt{1 + 4|\psi|^2} \right] \psi, \tag{2}
\]

where \( \Omega \equiv \omega_z/\omega_\perp \) is the normalized trap strength. In Eq. (2), the density \( |\psi|^2 \) is measured in units of \( a \), while length, time, and energy are measured, respectively, in units of the transverse harmonic oscillator length \( \alpha_\perp \equiv \sqrt{\hbar/m\omega_\perp} \), and \( \hbar \omega_\perp \).

Exact analytical dark-soliton solutions of Eq. (2) are not available. However, following the lines of Ref. [38] where the NLS equation with a generalized defocusing nonlinearity was considered, dark-soliton solutions can be found in an implicit
form (via a phase-plane analysis) or in an approximate form (via the small-amplitude approximation). On the other hand, exact analytical dark-soliton solutions are available in the above mentioned weakly interacting limit, and in the absence of the external potential, i.e., for the cubically nonlinear defocusing NLS model. A single dark-soliton solution on top of a background with constant density \( n = n_0 = \mu \) (with \( \mu \) being the chemical potential) has the form [1],

\[
\psi(z,t) = \sqrt{n_0} [i(v + B \tanh(\eta))] \exp(-i\mu t),
\tag{3}
\]

where \( \eta = \sqrt{n_0} B(z - \sqrt{n_0}vt) \), the parameter \( B \equiv \sqrt{1 - v^2} \) sets the soliton depth, \( \sqrt{n_0}B \), while the parameter \( v \) sets the soliton velocity, \( \sqrt{n_0}v \). Note that for \( v = 0 \) the dark soliton becomes a stationary kink (alias “black” soliton), while for \( v = 1 \) the dark-soliton solution (3) becomes the background solution. Multiple dark-soliton solutions are also available [1,20].

Below we will show that the above-mentioned instability of higher-order nonlinear modes occurs near the noninteracting limit but will cease to exist sufficiently deep inside the nonlinear regime (i.e., for sufficiently large condensates with large numbers of atoms \( N \)). In fact, considering both the second- and the third-order nonlinear modes of Eq. (2), we will demonstrate that they are initially (i.e., for a small number of atoms, sufficiently close to the linear limit) linearly unstable. Nevertheless, numerical continuation of the corresponding waveforms to larger values of \( N \) reveals a critical value of the number of atoms (depending on the anisotropy of the external harmonic trap) whereby, if exceeded, all the eigenfrequencies become real and, thus, the nonlinear stationary state becomes linearly stable.

III. DYNAMICS OF MULTIPLE DARK SOLITONS

A. Multiple soliton interactions: the homogeneous case

Let us consider, at first, the simplest possible multisoliton state, namely a pair of dark solitons located at \( z = \pm z_0 \) and moving with opposite velocities, i.e., \( v_1 = -v_2 = v \). Then, in the framework of the weakly interacting limit of Eq. (2) (and in the absence of an external potential), we may derive an equation for the trajectory of the soliton coordinate, \( z_0 \), as a function of time. This can be done on identifying the soliton coordinate as the location of the minimum density and using
the equation $\partial |\psi|^2/\partial z = 0$, with $\psi(z,t)$ given in Eq. (4), to obtain the result:

$$\cosh(2\sqrt{n_0}Bz_0) = \sqrt{\frac{n_0}{n_{\min}}} \cosh(T) - 2\sqrt{\frac{n_{\min}}{n_0}} \frac{1}{\cosh(T)} \quad (8)$$

(recall that $T = 2\sqrt{n_{\min}(n_0 - n_{\min})}$). Then, Eq. (8) can be used for the determination of the distance $z_0^*$ between the two solitons at the point of their closest proximity (corresponding to $t = 0$):

$$z_0^* = \frac{1}{2\sqrt{n_0} - \frac{n_0}{n_{\min}}} \cosh^{-1}\left(\frac{n_0}{n_{\min}} - 2\sqrt{\frac{n_{\min}}{n_0}}\right). \quad (9)$$

This equation holds for $n_{\min}/n_0 = v^2 < 1/4$, otherwise Eq. (9) provides a complex (unphysical) value for $z_0^*$. Physically, this means that there exists a critical value of the soliton velocity, namely $v_c = 1/2$, which separates two different regimes: in the first regime, “low-speed” solitons with $v < v_c$ are approaching each other and at $t = 0$ the wavefunction exhibits a single nonzero minimum (i.e., $\min(|\psi| = 0)$), thus low-speed solitons are reflected by each other (their lower kinetic energy is insufficient to overcome the interparticle repulsion). In the second regime, “high-speed” solitons with $v > v_c$ are approaching each other and at $t = 0$ the wavefunction exhibits a single nonzero minimum (i.e., $\min(|\psi| = 0)$) characterizing the location of both solitons; thus, high-speed solitons are transmitted through each other (their high kinetic energy overcomes the interparticle repulsion). Note that in the case of the critical velocity $v_c = 1/2$, the two-soliton wavefunction exhibits a single zero minimum at the collision point. The above results are also illustrated in Fig. 1, where different panels correspond to various collision scenarios. The above analysis underscores the fact that low-speed solitons are actually “well separated” solitons, in the sense that they can always be characterized by two individual density minima even at the collision point (the point of their closest proximity). On the contrary, the high-speed solitons completely overlap at the collision point and, thus, are not distinguishable during the collision. Thus, well-separated solitons appear to be reflected by each other and can safely be regarded as hard-sphere-like particles that interact through an effective repulsive potential (although, as we will see quantitatively, the description below will be surprisingly accurate even in the non-well-separated case).

To further elaborate on the above, let us consider the limiting case of extremely slow solitons, namely $n_0/n_{\min} = v^2 \ll 1/4$, for which the soliton separation is large for every time, i.e., the closest proximity distance is $z_0^* \gg 0$. In this case, the second term in the right-hand side of Eq. (8) is much smaller than the first one for every time (including $t = 0$) and can be ignored. This way, the soliton coordinate can be expressed as:

$$z_0 = \frac{1}{2\sqrt{n_0}B} \cosh^{-1} |v^{-1}\cosh(2n_0Bt)|, \quad (10)$$

which yields the soliton velocities:

$$\frac{dz_0}{dt} = \sqrt{\frac{n_0}{v^2 - 1}} \sinh(2n_0Bt) \cosh^{-1} |v^{-1}\cosh(2n_0Bt)|^{-1}. \quad (11)$$

The above equation shows that in the limit $t \to \pm \infty$, the soliton velocities take the asymptotic values $dz_0/dt = \pm \sqrt{n_0}v$, namely the values of the velocities of each individual soliton [see the definition of the single soliton velocity beneath Eq. (3)]. On the other hand, at $t = 0$, Eq. (11) yields $dz_0/dt = 0$; this means that as the dark solitons are approaching each other, they become slower, i.e., darker, and at $t = 0$ (corresponding to the point of their closest proximity) they become black, remaining at some distance away from each other. After such a head-on “black collision” [26], the dark solitons are reflected by each other and continue their motion in opposite directions.

We now proceed to determine the effective repulsive potential for well-separated dark solitons. This can be done by determining, at first, an equation of motion for the soliton coordinate: differentiating Eq. (10) twice with respect to time, and using Eq. (8) (without the second term, which is negligible for well-separated solitons), we obtain the equation of motion in the form $d^2z_0/dt^2 = -\partial V(z_0)/\partial z_0$, with the repulsive potential being given by:

$$V(z_0) = \frac{1}{2} \frac{n_0B^2}{\sinh^2(2\sqrt{n_0}Bz_0)}. \quad (12)$$

FIG. 1. The first and second panels show the soliton trajectories, depicted in the $z,t$ plane (here, $z$ and $t$ are respectively measured in units of the healing length $\xi = \sqrt{\hbar/(m\mu)}$ and $\hbar/(\mu t)$ for symmetric two-soliton collisions for different initial velocities: $v = 0.2$ (first panel) and $v = 0.8$ (second panel). In the former (latter) case the solitons are reflected by (transmitted through) each other. The third and fourth panels show, respectively, the soliton trajectories for an asymmetric two-soliton collision for initial velocities, $v_1 = 0.5$ and $v_2 = 0$, and for a three-soliton collision for initial velocities $v_1 = -v_2 = 0.4$ and $v_3 = 0$. In all panels, density plots of the wavefunction carrying the solitons as obtained by direct numerical integration of the homogeneous NLS equation are shown. The solid lines correspond to the solution of Eq. (14) (i.e., employing the effective interaction potential). In all cases the normalized density is $n_0 = 1$. 063604-4
It is worth noting here that since \( B = \sqrt{1 - v^2} \), the above potential is, in principle, a velocity-dependent one. Note that Eq. (12) recovers the result obtained in Ref. [25] (in that work, the sinh term in the denominator appears as a cosh term due to a typographical error [44]).

Although the potential of Eq. (12) is formally applicable only to symmetric collisions, it cannot nevertheless be applied also in the case of nonsymmetric collisions provided that an “average depth” of the two solitons is employed. In fact, it is possible to generalize this concept for an arbitrary number of solitons, \( n \): assuming that the \( i \)-th soliton (with \( i = 1, 2, \ldots, n \)) is characterized by a darkness \( B_i \), velocity \( v_i = \sqrt{1 - B_i^2} \), and a position \( z_i \), we may define the average depth \( B_{ij} = (1/2)(B_i + B_j) \) and the relative coordinate \( z_{ij} = (1/2)(z_i - z_j) \) for solitons \( i \) and \( j \), and express the interaction potential \( V_i \) in the presence of other solitons, as:

\[
V_i = \sum_{i \neq j} \frac{n_B B_{ij}^2}{2 \sinh^2[\sqrt{m_B} (z_i - z_j)]}.
\]  

(13)

Notice that Eq. (13) is reduced to Eq. (12) for \( v_i = -v_j = v \) (i.e., for \( B_i = B_j = B \)) and \( z_i - z_j = 2z_0 \).

Using Eq. (13), it is now straightforward to obtain equations of motion for a “lattice” consisting of an arbitrary number of dark solitons. Taking into regard that the Lagrangian \( L \) of \( n \) interacting solitons is \( L = T - V \), where \( T = \sum_i m_i (1/2) \dot{z}_i^2 \) (with \( \dot{z}_i \equiv \partial(z_i)/\partial t \)) and \( V = \sum_i V_i \) are the kinetic and potential energy, respectively, the Euler-Lagrange equations, \( d(\partial_t L)/dt - \partial_t L = 0 \), lead to the following set of dynamical evolution equations:

\[
\ddot{z}_i - \sum_k \left( \frac{\partial^2 V}{\partial \dot{z}_i \partial \dot{z}_k} \dot{z}_k^2 + \frac{\partial^2 V}{\partial \ddot{z}_i \partial \dot{z}_k} \right) + \frac{\partial V}{\partial z_i} = 0.
\]  

(14)

These \( n \) coupled equations of motion can then be used to calculate the trajectories \( z_i(t) \) of \( n \)-interacting dark solitons. It is worth pointing out here that in deriving Eq. (14), we have attempted to incorporate the character of the solitary waves as “deformable particles” with a velocity-dependent interaction potential [cf. Eq. (12) and related discussion]. This approximation will be tested a posteriori through the detailed comparison of the particle-based and the GP-based dynamical results.

We have performed systematic numerical simulations to investigate the range of validity of Eq. (14), both for cases of symmetric and asymmetric soliton collisions, as well as both for cases of low-speed (well-separated) and high-speed dark solitons. Various relevant examples are shown in Fig. 1. The simulations confirm that as long as the dark solitons are well separated from each other, i.e., if their depth (velocities) is (are) sufficiently large (small), their trajectories found by means of Eq. (14) almost coincide with the ones found by direct numerical integration of the NLS equation. This excellent agreement can be illustrated not only qualitatively but also quantitatively: this can be done by comparing the exact results for the collision-induced phase-shifts of the soliton trajectories to the ones found numerically by means of Eq. (14). In the case of two solitons, these phase shifts were calculated analytically in Ref. [1] and have the following form:

\[
\delta z_1 = \frac{1}{2 B_1} \ln \left( \frac{(v_1 - v_2)^2 + (B_1 + B_2)^2}{(v_1 - v_2)^2 + (B_1 - B_2)^2} \right),
\]  

(15)

\[
\delta z_2 = \frac{1}{2 B_2} \ln \left( \frac{(v_1 - v_2)^2 + (B_1 + B_2)^2}{(v_1 - v_2)^2 + (B_1 - B_2)^2} \right).
\]  

(16)

In Fig. 2 we compare the exact phase shifts provided by the above expressions to the ones determined by means of Eq. (14), which employ the effective repulsive potential of Eq. (12). Both cases of a symmetric (top panels of Fig. 1) and an asymmetric (bottom left panel of Fig. 1) collision are shown; it is clearly observed that the agreement between the two approaches is very good for soliton velocities \( v \leq 1/2 \), i.e., for well-separated dark solitons. Note that in the case of an asymmetric collision, \( v_1 \neq v_2 \), the two shifts are not equal, \( |\delta z_1| \neq |\delta z_2| \), while in the case of a symmetric collision, \( v_1 = -v_2 = v \) (and, thus, \( B_1 = B_2 = B \)), the phase shifts become equal, \( |\delta z_1| = |\delta z_2| = (2B)^{-1/2} \ln (1 + B^2/v^2) \).
B. Dynamics and interactions of multiple solitons in the trap

The above analysis of soliton interactions in the homogeneous case is of use in the inhomogeneous case as well. In particular, here we will consider multiple dark solitons in the presence of a harmonic trap, also taking into consideration that the condensate is cigar shaped, so the proper model is Eq. (2) (rather than its weakly interacting limiting case, i.e., the usual cubic NLS equation considered above). In such an experimentally relevant situation, we may employ the theoretical approach adopted in Ref. [18] and use an effective potential for dark solitons of the form:

\[ V_{\text{eff}} = V_{\text{eff}}(z_i) + V_i(z_i, \Delta z_i), \]

(17)

where \( V_i(z_i, \Delta z_i) \) is the interaction potential of Eq. (13) and the effective trapping potential \( V_{\text{ext}}^{\text{eff}} \) is given by:

\[ V_{\text{ext}}^{\text{eff}}(z_i) = \frac{1}{2} \omega_{\text{osc}}^2 z_i^2, \]

(18)

where \( \omega_{\text{osc}} \) is the oscillation frequency of a single dark soliton in the harmonic trap. The underlying assumption within this decomposition is that dark solitons are an effective particle moving under the combined influence of external forces from the confining potential and from the other solitons within the configuration. Each of these individual forces has an associated potential and hence the total force and associated motion stem from the combination of these potentials. A more subtle assumption is that while the effect of dimensionality on a single soliton is captured in the effective oscillation frequency \( \omega_{\text{osc}} \) (as will be discussed in detail below), the tail–tail interaction of the solitons is well approximated by its NLS counterpart. These assumptions will be validated a posteriori through our comparisons among theoretical, numerical, and experimental results below.

The oscillation frequency \( \omega_{\text{osc}} \) in Eq. (18) can readily be obtained by direct numerical integration of the mean-field model, i.e., Eq. (2). Being in the purely 1D regime, \( N\Omega\alpha/\omega_1 \ll 1 \) and simultaneously in the Thomas-Fermi limit \([6, 7]\) \((N\sqrt{2\Omega\alpha_1})^{1/3} \gg 1\), the soliton oscillation frequency is \( \omega_{\text{osc}} = \Omega/\sqrt{2} \), where \( \Omega \) is the normalized trap strength. This can be derived either by analyzing the dynamics of the dark soliton in the trap (see Ref. [30], as well as the review [9] and references therein) or by means of a BdG analysis [31]: in such a case, the oscillation frequency coincides with the lowest anomalous mode of the system \([6, 7]\). However, in the case under consideration (relatively small cigar-shaped BECs), the oscillation frequency differs from this asymptotic limit, taking values in the interval \( \Omega/\sqrt{2} < \omega_{\text{osc}} < \Omega \) due to deviation from the 1D regime \([32]\) and the fact that the TF limit in the longitudinal direction is not reached (see relevant earlier work in Ref. [45]). Furthermore, in the case under consideration, an additional dependence of the oscillation frequency on the oscillation amplitude was demonstrated numerically in Ref. [18], contrary to the situation in the Thomas-Fermi 1D limit, where the oscillation frequency does not depend on the soliton amplitude \([46]\).

Based on the above discussion, the interaction potential of Eq. (17) takes into account both the effective harmonic trap (including the dimensionality of the system), \( V_{\text{eff}}^{\text{ext}}(z_i) \), and the inter-soliton interaction potential, \( V_i(z_i, \Delta z_i) \) (derived for the homogeneous 1D regime). This potential has already been successfully used in Ref. [18], where the experimental findings for the symmetric collisions between two dark solitons were found to be in excellent agreement with the corresponding theoretical results. Here, we will show that the approach based on the use of the effective potential of Eq. (17) can also be generalized to the case of asymmetric collisions. Such a case can also be investigated experimentally in the context of the experimental setup of Ref. [18], where dark solitons are created by merging two condensates initially prepared in a double-well trap (see Sec. V).

In Fig. 3 (left) we show the dependence of the soliton oscillation frequency, \( \omega_{\text{osc}} \), on the oscillation amplitude for a one-, two-, and three-dark soliton configuration. As shown in this figure, the frequency has a relatively weak dependence on the oscillation amplitude (≈3.5% in the amplitude range

![Graph showing the soliton oscillation frequency as a function of the oscillation amplitude](image)

**FIG. 3.** (Color online) (Left) The soliton oscillation frequency, in units of \( v_s \), as a function of the oscillation amplitude, in units of \( \xi \), for a configuration of one (light gray curve), two (gray curves), and three dark solitons (black curves) (see right panel). Shown are the results obtained by the ordinary differential equations (ODEs), i.e., the equations of motion (14) with the potential in Eq. (17) for the two- and three-soliton states (solid lines), as well as results obtained by direct numerical integration of the partial differential equation (PDE) Eq. (2) (dashed lines). The normalized trap strength is \( \Omega \approx 0.06 \). (Right) Contour plot depicting the evolution of the density, according to Eq. (2), for the three-dark-soliton configuration, consisting of one stationary central soliton and two oscillating ones with a frequency \( \omega_{\text{osc}} \approx 0.85\Omega \). The thin solid (green) lines show the path of the density minima of all three solitons calculated by the equation of motion employing the effective potential of Eq. (17) (solid green line). The parameter values are \( N \approx 1700 \), \( \omega_s = 2\pi \times 53 \) Hz, \( \omega_1 = 2\pi \times 890 \) Hz (i.e., \( \Omega \approx 0.06 \)) and initial displacement from the trap center \( \delta z_0 = 10\xi \).
of \( \approx 18\xi \) for the one-soliton state, while this dependence becomes significantly stronger for the two- and three-soliton states (more than 20% in both cases). The left panel of Fig. 3 compares also results obtained by the equations of motion (14) with the potential in Eq. (17) (for the two- and three-soliton states) with results obtained by direct numerical integration of Eq. (2). For the numerical integration of the equations of motion we have used the following procedure. First, Eq. (2) is used to determine the amplitude dependence of the oscillation frequency for a single dark soliton with the given parameters (see the light gray curve in the right panel of Fig. 3). This dependence is then used to calculate the effective trap frequency \( \omega_{\text{osc}} \) for every oscillation amplitude in the two- and three-soliton configurations considered in Fig. 3. Finally, Eq. (14) are numerically integrated for every amplitude, using the effective interaction potential (17) and the aforementioned effective trap frequency, as well as the chemical potential (at the trap center) found in the framework of the 3D GP equation for the ground state of the condensate. The agreement between the results obtained by Eqs. (2) and (14) is found to be quite good.

In the right panel of Fig. 3 we show the evolution of the three-soliton state. Here, a sequence of asymmetric collisions between a stationary (with \( v_2 = 0 \)) dark soliton, located at the trap center, with a pair of oscillating solitons (with \( v_1 = -v_3 \)) in a cigar-shaped condensate (the trap strength is \( \Omega \approx 0.06 \)). The figure shows the evolution of the density, as obtained by direct numerical integration of Eq. (2), as well as the trajectories of the three solitons, as computed via Eq. (14) with the effective potential of Eq. (17) (note that each of the two moving dark solitons is the mirror image of the other one around \( z = 0 \), while the third one is, by symmetry, constrained to stay precisely at \( z = 0 \)). It is clear that the agreement between Eq. (2) and the reduced particle picture is fairly good: as concerns the oscillation frequency, the relative error between the numerical result \( v_{\text{osc}} = 0.85v_2 \) and the theoretical prediction \( v_{\text{osc}} \approx 0.83v_2 \) is \( \approx 2.5\% \). Nevertheless, we should mention that the observed disagreement after the third period of oscillation can safely be attributed to the strong emission of radiation (which occurs as early as at \( t = 0 \), due to the fact that the considered initial condition is not an exact solution of the model), as well as the excitation of the quadrupole mode of the condensate; in fact, the observed coupling of the soliton motion to internal degrees of freedom of the BEC is a purely nonlinear effect, induced by the large-amplitude oscillations of the solitons, that cannot be captured in the framework of our analytical approach (the latter relies on the assumption that soliton motion is decoupled from the dynamics of the background atomic cloud).

We complete the analysis of this section by studying analytically the case of dark solitons performing small oscillations around their equilibrium positions. In fact, we are going to use the Lagrangian approach devised above to connect the oscillation frequency obtained by the multisoliton dynamics to the eigenfrequencies of the anomalous modes of the stationary soliton states that will be obtained by a BdG analysis in the next section. In that respect, it is relevant to consider the simplest case of two well-separated solitons, which are assumed to be almost black (i.e., \( B_1 = B_2 \approx 1 \)). In such a case, the Lagrangian takes the form,

\[
\mathcal{L} = \frac{1}{2} \dot{z}_1^2 + \frac{1}{2} \dot{z}_2^2 - \frac{1}{2} \omega_{\text{osc}}^2 z_1^2 - \frac{1}{2} \omega_{\text{osc}}^2 z_2^2 - \frac{n_0}{\sinh^2[\sqrt{n_0}(z_2 - z_1)]},
\]

(19)

Then, using the Euler-Lagrange equations, and replacing the hyperbolic function \( \sinh \) by its exponential asymptote in the case under consideration, i.e., for \( |z_2 - z_1| \gg 0 \), the following equations of motion are obtained

\[
\ddot{z}_1 = -8n_0^{3/2}e^{-2\eta_0^2/\pi\sqrt{n_0}(z_2 - z_1)} - \omega_{\text{osc}}^2 z_1, \quad \ddot{z}_2 = 8n_0^{3/2}e^{-2\eta_0^2/\pi\sqrt{n_0}(z_2 - z_1)} - \omega_{\text{osc}}^2 z_2.
\]

(20)

The fixed points, \( Z_1, Z_2 \), of the above system can be easily found by setting the left-hand side equal to zero. The result is:

\[
Z = Z_2 = -Z_1 = \frac{1}{4\sqrt{n_0}} w \left( \frac{32n_0^2}{\omega_{\text{osc}}^2} \right),
\]

(21)

where \( \eta(w) \) is the Lambert’s \( \lambda \) function defined as the inverse of \( \eta(w) = we^w \). Then, considering small deviations \( (\eta_1, \eta_2) \) from the equilibrium positions \( Z_1, Z_2 \), we can Taylor expand the interaction potential keeping only the lowest-order term and, this way, derive the following linearized equations of motion:

\[
\ddot{\eta}_1 = 16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_2} (\eta_2 - \eta_1) - \omega_{\text{osc}}^2 \eta_1, \quad \ddot{\eta}_2 = -16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_1} (\eta_2 - \eta_1) - \omega_{\text{osc}}^2 \eta_2.
\]

(22)

Let us now consider the normal modes of the system and seek solutions of the form \( \eta_i = \eta_i e^{i\omega t} \), \( \omega = 1, 2 \), where \( \omega \) is the common oscillation frequency of both dark solitons. Then, substituting this ansatz into Eqs. (22), we rewrite the equations of motion as matrix eigenvalue equation, namely:

\[
-\omega^2 \eta = \begin{pmatrix}
-\omega_{\text{osc}}^2 - 16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_2} & 16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_2} \\
16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_1} & -\omega_{\text{osc}}^2 - 16n_0^2 e^{-4/\sqrt{n_0}\pi\eta_1}
\end{pmatrix} \eta.
\]

To this end, it is possible to obtain from the above system the characteristic frequency \( \omega_1 = \omega_{\text{osc}} \), which corresponds to in-phase oscillations of the two dark solitons, as well as the frequency \( \omega_2 \), which corresponds to out-of-phase oscillations of the two solitons. The latter is given by:

\[
\omega_2 = \sqrt{\frac{\omega_{\text{osc}}^2}{32n_0^2} + 32n_0^2 e^{-4/\sqrt{n_0}\pi\eta_2}}.
\]

(23)

The above procedure can also be applied to the case of three, almost black, solitons \( (B_i \approx 1, i = 1, 2, 3) \) considering only nearest-neighbor interactions. In this case, the equilibrium positions are given by the following expressions:

\[
Z_2 = 0, \quad Z_3 = -Z_4 = \frac{1}{2\sqrt{n_0}} w \left( \frac{16n_0^2}{\omega_{\text{osc}}^2} \right),
\]

(24)

while the three characteristic frequencies which correspond to the three normal modes of the system are the following,

\[
\omega_1 = \omega_{\text{osc}}, \quad \omega_2 = \sqrt{\omega_{\text{osc}}^2 + 16n_0^2 e^{-2/\sqrt{n_0}\pi\eta_2}}, \quad \omega_3 = \sqrt{\omega_{\text{osc}}^2 + 48n_0^2 e^{-2/\sqrt{n_0}\pi\eta_2}}.
\]

(25)
In the following section, we will elaborate on the investigation of the stability of the multiple soliton states and the derivation of their anomalous modes.

IV. STABILITY OF STATIONARY MULTI-SOLITON STATES

Having examined the location of the solitary waves, eigenmodes, and eigenfrequencies in an analytical form, we now turn to the numerical investigation of such stationary multisoliton states and to their corresponding BdG spectrum. We carry out the relevant computations first for the two-dark-soliton state and subsequently for the three-dark-soliton state.

A. The two-dark-soliton state

We start by considering the simplest possible stationary multisoliton state, namely the second-order nonlinear mode of Eq. (2). In the linear limit of $N \to 0$, this state corresponds to the second-excited state of the quantum harmonic oscillator. The excitation spectrum of this state contains a zero eigenvalue (corresponding to the Goldstone mode), two double eigenfrequencies located at $\Omega$ and $2\Omega$, as well as infinitely many simple eigenfrequencies. Below we will analyze the excitation spectrum in the nonlinear regime (i.e., when the number of atoms $N$ is increased) focusing, in particular, on the two double eigenvalue pairs mentioned above. Notice that in our numerical results (see below) we fix $\omega_\perp$ to the typical value $2\pi \times 400$ Hz.

First, we consider the two eigenfrequencies located at $\Omega$ (for $N = 0$) which, in the nonlinear regime, obtain opposite Krein signature. In particular, one of them has positive Krein signature and corresponds to the dipole mode [6,7], while the second one has negative Krein signature, i.e., the integral of the norm $\times$ energy product, $\int (|u|^2 - |v|^2) dx$ (in our units), is negative. In other words, in the nonlinear regime this eigenvalue becomes the eigenfrequency $\omega_1$ of one of the two anomalous modes of the system (recall that the number of anomalous modes in the excitation spectrum is the same as the number of dark solitons or nodes in the relevant waveform). Note that both eigenfrequencies originating from $\Omega$ in the linear limit [see dashed (solid) lines for the one with positive (negative) Krein signature in the top panel of Fig. 4] are real for every value of the number of atoms $N$, indicating the absence of any instability. In fact, the dipolar mode, per the relevant symmetry [6,7], remains fixed at $\omega = \Omega$, while the anomalous mode’s frequency corresponding to the single-dark-soliton mode is dependent on $N$.

Next, we consider the eigenvalue pair located at $2\Omega$ (for $N = 0$). Similarly to the previous case, these eigenfrequencies obtain opposite Krein signature: one eigenvalue has positive Krein signature and corresponds to the background condensate’s quadrupole mode [6,7], while the second one has negative Krein signature, thus being the eigenfrequency $\omega_2$ of the second anomalous mode of the system. An important difference from the previous case is that this second pair of double eigenfrequencies does become complex; see the bottom panel of Fig. 4, where the imaginary part of these eigenfrequencies is shown as a function of $N$. This implies that the respective nonlinear stationary state is unstable for sufficiently small atom numbers. Nevertheless, the instability occurs only near the linear limit (as was also predicted in Ref. [43]). However, as seen in the bottom panel of Fig. 4, when the number of atoms exceeds a critical value, namely $N = 438$ for a trap strength $\Omega = 0.1$, all the eigenfrequencies become real and the nonlinear state becomes linearly stable.

We have generally found that the larger the number of atoms, and the stronger the anisotropy of the harmonic trap, the more stable the configuration with the two stationary dark solitons is. For example, for $\Omega = 0.35$ this state is unstable up to the number of atoms $N = 1067$, while for $\Omega \approx 0.1$ the instability occurs for very small condensates, with number of atoms $N < 500$. Note that a similar behavior was also found in the framework of the 1D GP equation considered in Ref. [43] (results not shown here). Additionally, regarding the connection of our analysis with experimental observations, we note that within the parameter range of relevance to the recent experiment of Ref. [18], we found the two-dark-soliton state to be linearly stable.

Let us now return to the excitation spectrum of Fig. 4 and focus on the eigenfrequencies possessing negative Krein signature (see solid lines in the top panel of Fig. 4), namely the two anomalous modes. In the case of two dark solitons under consideration, the physical significance of these two anomalous modes has been discussed in Ref. [36]. More specifically, excitation of the anomalous mode with the smallest eigenfrequency, $\omega_2$, gives rise to an in-phase oscillation, i.e., the two dark solitons move toward the same direction without changing their relative spatial separation. On the other hand, excitation of the anomalous mode with the largest eigenfrequency, $\omega_1$, gives rise to an out-of-phase oscillation, i.e., the two dark solitons move in opposite
instability (for frequencies are identical to the eigenfrequencies of the first and second anomalous modes, respectively. (c) The manifestation of the dynamical modes. (a) and (b) The in-phase and out-of-phase oscillatory motion of the two dark solitons (for the two possibilities correspond, respectively, to \( \eta_{10} = \eta_{10} \) and \( \eta_{20} = -\eta_{10} \). The correspondence of the anomalous modes with the normal modes of the two-dark-soliton state is confirmed by direct numerical simulations. In particular, we have numerically integrated Eq. (2) with initial condition the nonlinear stationary mode excited by the corresponding anomalous modes, i.e., \( \psi(x; t = 0) = \psi_{DS}(x) + u(x) + v^*(x) \). The results (for parameter values \( \Omega = 0.1 \) and \( N \approx 1000 \)) are shown in Figs. 5(a) and 5(b). In Fig. 5(a) [Fig. 5(b)], the excitation of the first (second) anomalous mode results in an in-phase (out-of-phase) oscillatory motion of the two dark solitons, with the characteristic eigenfrequency of the first (second) anomalous mode, i.e., \( \omega_1 = 0.0784 \) (\( \omega_2 = 0.199 \)). Calculating numerically the maximum density of the stationary state and the equilibrium positions of the dark solitons, we find that \( n_0 = 0.623 \) and \( Z = 1.78 \pm 0.1 \). This allows us to make a comparison with the ones calculated analytically based on the theoretical approach of the previous section. Using Eq. (21), with \( \omega_{osc} = \omega_1 \), and Eq. (23) we find that \( Z = 1.85 \) and \( \omega_2 = 0.205 \), which differ only 3% from the corresponding numerically obtained values.

Finally, it is worth investigating the manifestation of the instability predicted above for small condensates, due the “collision” of the second anomalous mode with the quadrupole mode. In Fig. 5(c), we show the evolution of the density of a condensate with \( N \approx 400 \) confined in a trap with strength \( \Omega = 0.1 \) (as before). For these parameter values, the eigenfrequencies of the two anomalous modes are \( \omega_1 = 0.081 \) and \( \omega_2 = 0.188 + 0.0025i \). The numerical integration of Eq. (2) reveals that although the initial evolution of the density roughly follows the one observed in Fig. 5(b) (up to \( t \approx 700 \), the instability eventually manifests itself: the soliton motion excites the quadrupole mode of the system, and this excitation results in a breathing behavior of the BEC [see Fig. 5(c)].

B. The three-dark-soliton state

Let us now consider the third-order nonlinear mode of Eq. (2). In the linear limit, the excitation spectrum of this state consists of a zero eigenvalue, three double eigenfrequencies located at \( \Omega \), \( 2\Omega \), and \( 3\Omega \), as well as infinitely many simple ones. In the nonlinear regime, and similarly to the previous case, each of the three aforementioned pairs leads to opposite Krein signature modes. In Fig. 6 we show the real [Fig. 6 (top)] and imaginary [Fig. 6 (bottom)] parts of the lowest eigenfrequencies as a function of the number of atoms \( N \) (for a trap strength \( \Omega = 0.1 \) as before). In Fig. 6 (top), the eigenfrequencies depicted by dashed (solid) lines correspond to ones with with positive (negative) Krein signature. Regarding the ones with positive Krein signature, we note that the lowest ones, starting from \( \omega_1 = 0.1 \) and \( \omega_2 = 0.2 \) for \( N = 0 \), correspond to the dipole and quadrupole modes, respectively. As before, the system’s ability to sustain dipolar oscillations for all \( N \) with a frequency \( \Omega \) preserves the dipolar frequency at \( \omega_1 = 0.1 \) throughout the relevant figure and precludes the possibility of a quartet-inducing collision with the lowest in-phase-oscillation anomalous mode of the system.
We now focus on the two upper double pairs (and their anomalous modes), located at 2Ω and 3Ω in the linear limit. These become complex in the nonlinear regime. As is observed in Fig. 6, the upper pair (starting from 3Ω for \( N = 0 \)) splits into real eigenfrequencies at very small values of the number of atoms \( N \), while the lower pair (starting from 2Ω at \( N = 0 \)) remains complex for larger values of \( N \). For the assumed trap strength \( \Omega = 0.1 \), the complex eigenfrequencies become real beyond the critical value of \( N \approx 880 \) and, thus, the nonlinear mode becomes dynamically stable. Here, it is worth mentioning that this state remains stable for the values of \( N \gtrsim 880 \) considered herein, although at \( N \approx 1395 \) another collision appears: indeed, at a point marked by a circle in the top panel of Fig. 6, the eigenfrequencies starting from \( \omega_r = 0.3 \) and \( \omega_r = 0.4 \) for \( N = 0 \), which possess opposite Krein signature, cross each other. Nevertheless, this collision does not lead to instability because the eigenmodes associated with these eigenfrequencies remain orthogonal at the collision point. This happens due to the opposite parity of the colliding eigenmodes [43].

So far, to investigate the stability of the three-dark-soliton state (as well as the two-soliton state considered in the previous section) we kept the trap strength \( \Omega \) fixed and varied the number of atoms \( N \). It is also worthwhile (and experimentally relevant) to reverse the procedure, i.e., to keep the number of atoms fixed, at \( N = 1000 \), and vary the harmonic trap strength \( \Omega \), as shown in Fig. 7. In this figure, it is readily observed that the nonlinear state remains stable up to the critical value \( \Omega = 0.12 \). Beyond this value, the eigenfrequency of the second anomalous mode collides with that of the quadrupole mode and becomes complex. Typical examples of the spectral plane, for both stable and unstable cases, are shown in the insets of Fig. 7. For the stable case of \( \Omega = 0.05 \), the eigenfrequencies of the anomalous modes are found to be \( \omega_1 = 0.039 \), \( \omega_2 = 0.0988 \), and \( \omega_3 = 0.1626 \), while for the unstable case of \( \Omega = 0.18 \) the respective values are \( \omega_1 = 0.1489 \) and \( \omega_2 = 0.5994 \); note that there exist also two complex eigenfrequencies at \( \omega_2 = 0.3379 \pm 0.01i \), stemming from the above-mentioned collision. Once again, we calculate numerically the maximum density of the state, \( n_0 = 0.3817 \), and the equilibrium positions of the dark solitons: \( Z_2 = 0, Z_3 = -Z_1 = 4.3 \pm 0.1 \). Using Eq. (24), with \( \omega_{osc} = \omega_1 \), we find that \( Z = 4.54 \), in good agreement with the numerically obtained, while the remaining two normal mode frequencies given by Eqs. (26) and (27) are found to be \( \omega_2 = 0.1, \omega_3 = 0.1647 \) and differ by less than 2% from the ones obtained by the BdG analysis.

Similarly to the two-soliton state, we now present results of direct numerical integration of Eq. (2) with an initial condition given by the third nonlinear mode excited by the corresponding anomalous modes. First, we study the stable case, with \( \Omega = 0.05 \) and \( N = 1000 \), and then the unstable case, with \( \Omega = 0.18 \) and \( N = 1000 \). In the stable case, when the first anomalous mode is excited, the three dark solitons perform an in-phase oscillation with the characteristic eigenfrequency of the corresponding anomalous mode; see Fig. 8(a). The excitation of the second anomalous mode results in the following configuration: the two outer solitons are moving in opposite directions (symmetrically around \( z = 0 \)), while the center soliton is a stationary one; see Fig. 8(b) and also pertinent experimental result in Fig. 10(b) below. Note that, in this case, the outer solitons reverse directions rapidly when they collide, creating a nonsinusoidal pattern. On the other hand, when the third anomalous mode is excited, the two outer solitons are oscillating in-phase while the center soliton is oscillating out-of-phase with respect to the outer ones; see Fig. 8(c). In this case, the outer solitons collide with the center soliton and, after each collision, the center soliton reverses direction. Finally, in the unstable case, we use as an initial condition the third nonlinear state excited by the mode associated with the complex eigenfrequencies. As seen in Fig. 8(d), initially the outer dark solitons are moving out of phase, following the configuration observed in Fig. 8(b). Nevertheless, similarly to the two-dark-soliton state, the motion of the solitons gradually excites the quadrupole
FIG. 8. (Color online) Spatiotemporal evolution of the condensate density, in dimensionless units, after the excitation of the three anomalous modes. 

mode of the system, resulting in a breathing behavior of the condensate, signaling the manifestation of the relevant dynamical instability. Clearly, in such a case, the oscillation amplitude of the dark solitons is not constant anymore.

V. EXPERIMENTAL CREATION OF MULTIPLE ATOMIC DARK SOLITONS

Dark solitons can be created experimentally, e.g., by the method of matter-wave interference, which can be considered as a form of density engineering [18,19]. This method makes use of the fact that an interference pattern in the presence of interatomic interactions may generate a train of dark solitons [47,48]. Our experimental realization of density engineering involves two BECs, initially prepared in a double-well potential. The latter is created by the superposition of a crossed optical dipole trap and a one-dimensional optical lattice [49]. Removing the optical lattice leads to the merger of the two initially separated BECs in the harmonic trap.

To get a deeper insight into the physics of the matter-wave interference process, we follow the arguments of Ref. [48] (see also recent work in Ref. [19]) and start our considerations from the noninteracting (linear) regime. In this regime, the total wavefunction of the system can be well approximated by a linear superposition of the wavefunctions that each individual BEC would have, i.e., a superposition of the harmonic oscillator eigenmodes. The two initially well-separated BECs interfere at the trap center, produce a linear interference pattern, and then separate again regaining their initial shape. In this noninteracting (linear) case the initial distance $D$ between the two BECs exceeds a certain critical distance $D_c$ [48] (see also Ref. [18]) so the kinetic energy exceeds the peak nonlinear energy (at the center of the fringes). In the opposite case, i.e., for $D < D_c$, the system enters in the interacting (nonlinear) regime, where the interference pattern consists of stable fringes with a phase jump of the order of $\pi$ across them; such fringes can naturally be identified as genuine dark solitons [48]. Notice that in the nonlinear regime, the initially distinct condensates, instead of reforming as separate objects, form a combined condensate undergoing a quadrupole oscillation.

At this point, it is important to note that the above physical picture can be connected to the considerations of the theoretical part of this work presented in Sec. IV as follows. The linear interference pattern can be considered as a time-dependent superposition of harmonic oscillator eigenmodes. In the nonlinear regime, the interference pattern consists of a chain of moving dark solitons; the respective stationary multisoliton state can be regarded as the nonlinear counterpart of an excited state of the harmonic oscillator, while the chain of the moving dark solitons is a superposition of the aforementioned nonlinear counterparts of the respective (linear) excited states. Thus, the descriptions originating either from the noninteracting limit of a single BEC or the interference process of two BECs are actually the same.

Our experiments are conducted following the procedure described in Ref. [18], where the observation of two oscillating
and colliding solitons was reported. In the following, we will extend this scheme to the preparation of a single and of multiple dark solitons in a harmonic trap. Before proceeding to the presentation of our results, it is worth mentioning the following: in our experiments the initial distance $D$ between the BEC fragments is four times smaller than the above mentioned critical distance $D_c$; furthermore, we note that the initial trap frequencies were ramped down, with ramping times chosen so as to minimize the excitation of the quadrupole mode.

A. Controlling the created macroscopically excited state

The fringe spacing of a matter-wave interference pattern depends on the momentum of the two merging atom clouds. Therefore, it is possible to vary the number of created solitons in this process by controlling the relative velocity. An increase of the relative velocity between the atom clouds lowers the final value of the longitudinal trapping frequency $\nu_z$ by controlling the relative velocity. An increase of the relative velocity between the atom clouds lowers the in this process by controlling the relative velocity. An increase of the relative velocity between the atom clouds lowers the.

![Image](https://example.com/image1.png)

**FIG. 9.** Increasing the distance between the created solitons can be realized by increasing the ramping-down time of the optical lattice $\tau_{OL}$, as shown by numerical simulations of Eq. (2): (a) $\tau_{OL} = 0$ ms, (b) $\tau_{OL} = 2$ ms, (c) $\tau_{OL} = 4$ ms.

A matter-wave interference pattern depends on the relative phase difference $\Delta\phi$ of the two merging atom clouds. If $\Delta\phi = 0$ a symmetric pattern with an even number of solitons is produced. A small phase difference leads to an asymmetric evolution pattern of the created solitons, while a phase difference close to $\pi$ leads to the creation of an odd number of solitons. Such phase difference can be created by changing the symmetry of the potential, which results in an energy difference between the levels of the two wells of the double-well potential. Maintaining this asymmetry for a certain hold time accumulates a phase difference between the two BECs. In a simple approximation, the phase difference is given by $\Delta\phi \approx \Delta\mu / \hbar \cdot t$. By adapting the asymmetry and the time of phase accumulation before releasing the two condensates from the double well, arbitrary phase differences can be achieved. Especially interesting is the case were the initial phase difference is exactly $\pi$. Shifting the symmetry of the initial double-well potential experimentally is realized by shifting the second beam of the dipole trap with respect to the optical lattice.

![Image](https://example.com/image2.png)

**FIG. 10.** (Color online) (a) Observation of the oscillation of four dark solitons including the creation process. The evolution is averaged over 10 experimental runs. The point in time where the creation process of the solitons is finished is marked by the dashed line. (b) Experimental observation of three dark solitons in a harmonic trap averaged over 16 runs. The creation process of the solitons is not shown in this case. The soliton in the center of the trap is at rest (black soliton) whereas the two outer ones oscillate. The time evolution plots were obtained by integrating the images over their transverse axis, meaning that each vertical line shows the longitudinal density of the BEC at a certain point in time.

The successful experimental realization of four solitons plus two additional weak ones with extremely high oscillation amplitude is shown in Fig. 10(a). The illustrated oscillation dynamics, recorded with a time resolution of 1 ms, includes the creation process of the solitons starting at the initial double-well potential and including the ramping down of the optical lattice. The creation process ends at the dashed line, where the final value of the longitudinal trapping frequency is reached after a suitable ramping down from the value necessary to obtain the double-well potential. Note that to produce the experimental images (Figs. 10 and 11 below) the atomic density, after a certain evolution time in the harmonic trap, is obtained using standard absorption imaging with an optical resolution of approximately 1 $\mu$m. A short time of flight on the order of 1 ms is used to enhance the contrast.

A matter-wave interference pattern depends on the relative phase difference $\Delta\phi$ of the two merging atom clouds. If $\Delta\phi = 0$ a symmetric pattern with an even number of solitons is produced. A small phase difference leads to an asymmetric evolution pattern of the created solitons, while a phase difference close to $\pi$ leads to the creation of an odd number of solitons. Such phase difference can be created by changing the symmetry of the potential, which results in an energy difference between the levels of the two wells of the double-well potential. Maintaining this asymmetry for a certain hold time accumulates a phase difference between the two BECs. In a simple approximation, the phase difference is given by $\Delta\phi \approx \Delta\mu / \hbar \cdot t$. By adapting the asymmetry and the time of phase accumulation before releasing the two condensates from the double well, arbitrary phase differences can be achieved. Especially interesting is the case were the initial phase difference is exactly $\pi$. Shifting the symmetry of the initial double-well potential experimentally is realized by shifting the second beam of the dipole trap with respect to the optical lattice.

Figure 10(b) shows the experimental realization of three dark solitons in a harmonic trap created by the above discussed method. The trap frequencies used in the experiment are $\nu_z = 36.1 \pm 0.25$ Hz (longitudinal frequency), $\nu_\perp = 407.5 \pm 40.8$ Hz (transverse frequency). The mean number of atoms in the BEC is $N = 1570 \pm 150$. In this measurement the height of the optical lattice is ramped down linearly on a time scale of $\tau_{OL} = 2$ ms. The final value of the longitudinal trapping frequency $\nu_z$ is reached after ramping down within.
7 ms from the value of $\nu_{\text{initial}} = 63$ Hz necessary for obtaining the double-well potential. In the performed experiment the oscillation amplitude of the two outer solitons, $A_{\text{osc}} = (21 \pm 0.6)\xi$, is relatively large. Therefore, the oscillation frequency, $\nu_d$, is only moderately increased compared to the one of the single soliton case due to the soliton-soliton interaction: in the three-soliton case we find that $\nu_d/\nu_z = 0.775 \pm 0.006$, while the frequency value for the single soliton case (with the same background characteristics and soliton amplitude), according to numerical simulation, is found to be $\nu_d/\nu_z \approx 0.73$.

Our method should offer the possibility of creating a single, stationary soliton, corresponding to the first excited state of a harmonic trap [47]. Numerical simulations reveal that this can be achieved by increasing the ramping-down time of the optical lattice $\tau_{OL}$ further, which decreases the kinetic energy of the collision process. In the lowest collisional state only one interference fringe is produced which produces a single dark soliton (see right panel of Fig. 11). However, due to experimental problems, the single (moving) dark solitons that were created by this method were fluctuating in position from shot to shot.

The above results illustrate the possibility of experimentally generating not only a single pair of dark solitons as in Ref. [18] but rather of an essentially arbitrary number of such solitons.

B. Comparison between experiments and theory

In this section, we present a comparison between experimental observations for the two- and three-soliton configurations and numerical results obtained by direct numerical integration of the mean-field model, Eq. (2). Particularly, we present experimental results that have been obtained for the two-soliton configuration with three different normalized trap strengths, namely $\Omega \approx 0.06$, $\Omega \approx 0.09$, and $\Omega \approx 0.14$, and a sole experimental result for the three-soliton configuration, obtained for $\Omega \approx 0.09$. It is also relevant to mention that, for all experimental results, the dimensionality parameter takes values $N\Omega a/a_\perp \approx 1.2-1.8$, which shows that the experiments are indeed conducted in the dimensionality crossover regime between 1D and 3D; this fact ensures the dynamical stability of the dark solitons and justifies the use of Eq. (2) as a proper mean-field model. For a detailed comparison between the experimental results presented in this work (as well as the ones presented in Ref. [18]) and other matter-wave dark-soliton experiments, see Ref. [50].

The experimental and numerical results are compared in Fig. 12, where we show the soliton oscillation frequency as a function of the oscillation amplitude. For the two-soliton state, the gray shaded area depicts the range of oscillation frequencies and amplitudes obtained by the numerical simulations for the parameter sets used in the experiments; different symbols are used for the three different trap strengths, namely circles for $\Omega \approx 0.06$, diamonds for $\Omega \approx 0.09$, and squares for $\Omega \approx 0.14$. For the three-soliton state, there exists only one measurement (for $\Omega \approx 0.09$), which is compared to the corresponding theoretical prediction (dashed line). The experimental errors are estimated from the standard deviation of the calibrations of the trap frequencies and the systematic error of the number of atoms.

As far as the results pertaining to the three-soliton configuration, it is worth mentioning the following. In the experiment, the outer solitons were created with a rather high oscillation.
amplitude which results in strong emission of radiation (in the form of sound waves) and the excitation of the quadrupole oscillation (see also the simulation shown in the right panel of Fig. 3, where these effects are already observed with a smaller oscillation amplitude). In such an extreme case, where the soliton performs a nonharmonic motion exciting normal modes of the condensate, a discrepancy of the order of $\approx 3.5\%$ (see Fig. 12) seems to be legitimate. Furthermore, we note that—for the same reasons—we cannot compare the above experimental result, or the numerical result obtained in the framework of Eq. (2), with our analytical estimations [cf. Eq. (26)], as the latter were based on the assumption of small-amplitude oscillations of the solitons.

VI. CONCLUSIONS

In the present work, we attempted to quantify the existence, stability, and dynamics of multiple atomic dark solitons by examining in detail the prototypical cases of two- and three-dark-soliton states. We provided two complementary viewpoints corroborating the same basic picture. A first approach was that of considering the solitons as particles, which interact with each other through an exponential tail-tail interaction and are confined within a parabolic trap (appropriately incorporating the effect of dimensionality). This particle picture provided us with a detailed understanding of the repulsive nature of the intersoliton interaction and its implication on collision-phenomena and on how it can be combined with the restoring force of the parabolic confinement to provide for effective stationary states of the system (i.e., of the dark-soliton “crystal”). Within the context of these equations of motion the normal modes of this crystal were also examined and were associated with relative motions between the solitary waves. The second viewpoint came from the consideration of an effective quasi-one-dimensional partial differential equation (which incorporates appropriately the transverse confinement of the cloud) and starting from the linear limit of the number of atoms $N \rightarrow 0$ which has the well-known quantum harmonic oscillator eigenfunctions and developing the multi-dark-soliton states as natural continuations of appropriate (second- or third- or higher-) excited modes of the linear problem. In that context, the excitation spectrum contained the modes of the background BEC (omitted from the particle picture), as well as the anomalous modes pertaining to the dark-soliton quasiparticles, which were, in turn, associated with the above mentioned normal modes. These two approaches together with a detailed understanding of the experimental setup of Ref. [18] provide key insights on what types of modes can be excited in the experiment, what intrinsic frequencies should be associated with them, and, furthermore, what types of instabilities and resonances with background excitations these modes can be expected to induce.

One of the future directions of the present program would be to generalize this picture to the extent possible to the $n$-dark-soliton lattice, formulating and addressing questions about the characterization of the normal modes of such a “dark-soliton-crystal,” as well as questions about the conditions under which this crystal could potentially undergo phase transitions, possibly to a state such as a “dark-soliton-gas.” On the other hand, another natural generalization of the present program would be that of considering a quasi-two-dimensional analog of the waveforms and the corresponding stability and dynamics namely that of multivortex structures and the associated particle picture. Studies along these directions are presently in progress and will be reported in future publications.

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