Stability of gap solitons in a Bose-Einstein condensate

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We analyze the dynamical stability of gap solitons formed in a quasi-one-dimensional Bose-Einstein condensate in an optical lattice. Using two different numerical methods we show that, under realistic assumptions for experimental parameters, a gap soliton is stable only in a truly one-dimensional situation. In two and three dimensions, resonant transverse excitations lead to dynamical instability. The time scale of the decay is numerically calculated and shown to be large compared to the characteristic time scale of solitons for realistic physical parameters.

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I. INTRODUCTION

One of the most fundamental facts about Bose-Einstein condensates (BEC) of dilute atomic gases is that they can be very well described by the Gross-Pitaevskii equation (GPE)

$$i\hbar\,\dot{\psi}(\mathbf{x},t)=\left(\frac{\mathbf{p}^2}{2M}+V(\mathbf{x})+\kappa|\psi(\mathbf{x},t)|^2\right)\psi(\mathbf{x},t), \qquad (1)$$

where ψ denotes the collective wave function of condensed atoms with mass M and coupling constant $\kappa := 4\pi\hbar^2 a N_A/M$, with a being the s-wave scattering length and N_A denoting the number of atoms in the BEC (see, e.g., Ref. [1]). Therefore a BEC provides a physical realization of many nonlinear wave phenomena among which the formation of solitons is particularly interesting.

For our purposes solitons are wave packets in which the dispersive effect of the kinetic term is exactly canceled by the nonlinear interaction energy so that their shape does not change. Two fundamental types of solitons have so far been experimentally realized in a BEC: dark solitons [2,3] correspond to a stable density dip in a BEC of repulsive atoms. Bright solitons do exist for atoms with attractive interaction ($\kappa<0$) and are described by a one-dimensional solution of the GPE for vanishing potential $V(\mathbf{x})$,

$$\psi_{\text{bright}}(z,t)=\frac{1}{\sqrt{2\text{w}}}\operatorname{sech}(z/w)\,e^{-i\omega_s t}, \qquad (2)$$

where $w=2\hbar^2/(M\kappa_{1D})$ and $\omega_s=\hbar/(2Mw^2)$. To relate the one-dimensional GPE to its three-dimensional origin, we assumed that the BEC is tightly trapped in the transverse direction so that no transverse excitations can occur. Taking the BEC to be in the transverse ground state amounts to replacing the three-dimensional coupling constant κ by $\kappa_{1D}=\kappa/A_\perp$, where $A_\perp=2\pi a_\perp^2$ and with $a_\perp:=\sqrt{\hbar/(M\omega_\perp)}$ being the harmonic oscillator length of the transverse potential. The attraction between the atoms prevents the dispersion of the sech-shaped wave packet. Note that for $\kappa_{1D}>0$, corresponding to repulsive atom-atom interaction, the kinetic energy and the interaction energy have the same sign and therefore cannot cancel each other, a bright soliton is then not possible. Very recently bright solitons have been created in a quasi-one-dimensional setup [4,5], where the transverse potential $V(x,y)$ tightly confines the BEC, thus suppressing transverse excitations and three-dimensional collapse.

In this paper we are concerned with the dynamical stability of gap solitons. This collective state, which has not yet been realized experimentally, exists for repulsive atoms ($\kappa_{1D}>0$) in a periodic potential $V(z)$ and is related to bright solitons. The basic idea of a gap soliton is the following: as is well known the energy eigenvalues of noninteracting particles in a periodic potential are given by energy bands $E_n(q)$, where n is the band index and q denotes the quasimomentum. If a particle's state is prepared in the lowest band only the dispersion relation $p^2/(2M)$ in free space is replaced by the lowest band energy $E_0(q)$. Around the upper band edge, which we take to be at $q=0$, this energy can be approximated by $E_0(q)\approx E_0(0)+q^2/(2M^*)$, where $M^* := [d^2E_0(q)/dq^2]^{-1}|_{q=0}$ is the effective mass of the particle in the periodic potential. Since at the upper band edge $M^*<0$ the "kinetic energy" becomes negative and a cancellation with the positive interaction energy becomes possible. The corresponding state is called a *gap soliton*.

Gap solitons have been realized in nonlinear optics using a periodic modulation of the propagation medium [6]. In nonlinear atom optics they have first been predicted by Lenz *et al.* [7] in the context of light-induced nonlinearities [8,9]. Here we are concerned with the collision-induced nonlinearity appearing in Eq. (1).

We consider a BEC that is placed in a one-dimensional optical lattice, created by far-detuned laser light with wave number k_L, which produces an optical potential of the form $V(z)=-V_0\cos(2k_L z)$, and is subject to a tight harmonic transverse confinement of the form $V_\perp(x,y)=M\omega_\perp^2(x^2+y^2)/2$. Here $\omega_\perp$ is the transverse trap frequency and V_0 is the strength of the optical potential. The derivation of the corresponding gap soliton solution of the one-dimensional GPE is tedious and includes a multiple scales analysis. For nonlinear optics it has been derived by Sipe and co-workers [10]. For nonlinear atom optics a related derivation has been sketched in Refs. [7,11,12]. One finds that, within the effective-mass approximation, the one-dimensional gap soliton is described by

$$\psi_{\mathrm{gap}}(z,t)=\varphi_{\mathrm{be}}(z,t)\frac{1}{\sqrt{2\tilde{w}}}\,\mathrm{sech}(z/\tilde{w})e^{-i\tilde{\omega}_s t}, \tag{3}$$

where $\varphi_{\mathrm{be}}(z,t)=\varphi_{\mathrm{be}}(z)e^{-iE_0(0)t/\hbar}$ is the solution of the linear Schrödinger equation that corresponds to the upper band edge. The parameters $\tilde{w}$ and $\tilde{\omega}_s$ have the same form as w and ω for the bright soliton but with M replaced by $|M^*|$ and κ_{1D} replaced by $\tilde{\kappa}_{1D}:=\kappa_{1D}\int|\varphi_{\mathrm{be}}(z)|^4dz$. Apart from φ_{be} solution (3) just corresponds to a bright soliton for a particle of mass $|M^*|$ and coupling constant $\tilde{\kappa}$.

The range of experimentally promising values for the optical potential strength V_0, the number of condensed atoms N_A, and the transverse confinement frequency $\omega_\perp$ has been examined in a study by Brezger *et al.* [13]. The number of atoms necessary for the generation of a first-order soliton can be estimated by comparing the time scales of dispersion $T_d=m^*w^2/\hbar$ and the nonlinearity $T_{nl}=\hbar/\tilde{\kappa}_{1D}|\Psi|^2_{max}$, where $|\Psi|^2_{max}$ is the absolute value squared of the wave function at the center of the wave packet [14]. Experimentally realistic parameters are $w=10\ \mu$m (BEC released from a TOP trap), an effective mass of $m^*=0.2$ m (corresponding to $V_0\approx=E_{rec}=\hbar^2k_L^2/(2M)$) and a transverse confinement frequency of $\omega_\perp=110\ \mathrm{s}^{-1}$. By comparing the given time scales we expect that $N_A\approx400$ atoms lead to the formation of a soliton. In the following we will focus on this case. We will also consider the effect of a variation of N_A which allows to test the quasi-one-dimensionality of the gap soliton.

II. STABILITY THEORY OF GAP SOLITONS AND BRIGHT SOLITONS

A stationary solution ψ_0 of the GPE, with chemical potential μ, is called *dynamically stable* if a small deviation $\delta\psi$ from ψ_0 will not grow with time. In this case a small perturbation will not cause the solution to evolve into a completely different wave packet. To study dynamical instability one can either directly integrate the GPE or solve the associated Bogoliubov-de Gennes equations (BDGE, see, e.g., Ref. [15]). The latter arise when one writes the wave function $\psi(\mathbf{x},t)$ in the form

$$\psi(\mathbf{x},t)=\exp(-i\mu t/\hbar)[\psi_0(\mathbf{x})+\delta\psi(\mathbf{x},t)], \tag{4}$$

and linearizes the GPE in $\delta\psi$. By making the ansatz of a stationary perturbation,

$$\delta\psi(\mathbf{x},t)=u(\mathbf{x})\exp(-i\omega t)-v^*(\mathbf{x})\exp(i\omega t), \tag{5}$$

one arrives at the BDGE

$$\hbar\omega u=\mathcal{L}u-\kappa\psi_0^2 v,$$

$$-\hbar\omega v=\mathcal{L}v-\kappa(\psi_0^*)^2u, \tag{6}$$

with $\mathcal{L}:=\mathbf{p}^2/(2M)+V(\mathbf{x})-\mu+2\kappa|\psi_0|^2$. A solution (u,v) with eigenvalue ω corresponds to a quasiparticle mode. The set of all ω forms the quasiparticle spectrum which in general is complex. One can show [1] that if ω is in the spectrum then so is ω^*. Using Eq. (5) it is seen that the existence of a nonzero imaginary part of one quasiparticle frequency implies exponential growth of the mode and hence dynamical instability of the state ψ_0. To demonstrate that the gap soliton is stable, therefore amounts in showing that the associated quasiparticle spectrum is real.

III. NUMERICAL METHODS

Although the direct numerical integration of the GPE is easy to implement using the split-step method [16] it has the disadvantage of not being practical for a 3D study of the gap soliton. The reason is that state (3) includes two very different spatial scales: the laser wavelength $2\pi/k_L$ and the width $\tilde{w}$ of the soliton's envelope. To cover both scales simultaneously it was necessary to consider at least 260 periods of the optical potential or 2000 spatial points along the z axis. Since the number of total points in 3D is restricted by the capacity of the computer, a 3D simulation of a gap soliton becomes impractical. We therefore have applied this method only to 1D and 2D simulations.

To derive the spectrum of quasiparticles around the gap soliton we followed the method of Ref. [15] and expanded the modes (u,v) of Eq. (6) as well as the gap soliton wave function ψ_0 in a set of basis functions. For the z direction we have chosen a number of n_z Bloch wave functions which are eigenstates of the linear Schrödinger equation with potential $V(z)=-V_0\cos(2k_Lz)$. As transverse modes we used n_xn_y harmonic oscillator eigenstates corresponding to the transverse trapping potential. This turns Eq. (6) into an eigenvalue problem for a $2n\times2n$ matrix, where $n:=n_xn_yn_z$ is the total number of mode functions. However, before this eigenvalue problem can be solved, one first has to find the exact wavefunction of the gap soliton in the given set of basis modes (in practice a reasonably large subset is sufficient).

Since analytical solution (3) is only approximately correct a stability analysis will inevitably lead to a complex quasiparticle spectrum because of the finite difference to the exact solution. To find the exact solution ψ_{exact}, we have used a self-consistent field approach (SCF): after the expansion of the GPE in the basis modes Eq. (1) is turned into a set of coupled nonlinear algebraic equations for the expansion coefficients of ψ_0. Using approximate solution (3) as an ansatz $\psi_{\mathrm{trial,1}}$ we insert it into the GPE to evaluate the nonlinear terms. The resulting equation,

$$E\psi(\mathbf{x})=\left(\frac{\mathbf{p}^2}{2M}+V(\mathbf{x})+\kappa|\psi_{\mathrm{trial}}(\mathbf{x})|^2\right)\psi(\mathbf{x}), \tag{7}$$

represents a linear eigenvalue problem for ψ and can easily be solved using standard numerical methods. We pick that solution $\psi_{\mathrm{trial,2}}$ out of all eigenstates of Eq. (7) which has the least deviation from our previous guess $\psi_{\mathrm{trial,1}}$ and iterate this procedure until the change in the trial wave function is below a given value (we used a relative change of 10^{-14} as accuracy goal). The converged wave function may then correspond to the true gap soliton.

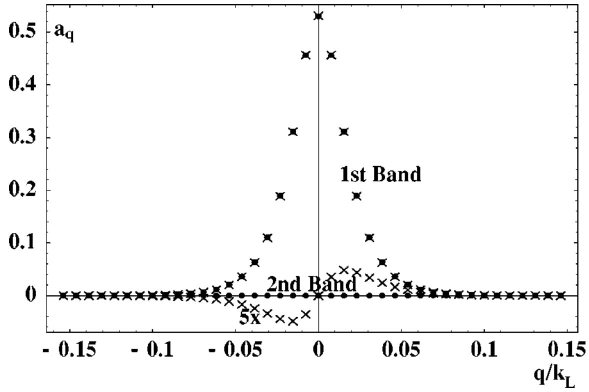

FIG. 1. The expansion coefficients $a_q = \langle \varphi_q(z) | \psi(z) \rangle$ of the wave function ψ in the basis of Bloch functions φ_n around the band edge of the first and second bands. The crosses indicate the initial wave function and the dots indicate the converged wave function. The coefficients of the first band are nearly identical. The coefficients of the second band are all close to zero. The coefficients of the initial wave function in the second band have been multiplied by 5 to improve the presentation.

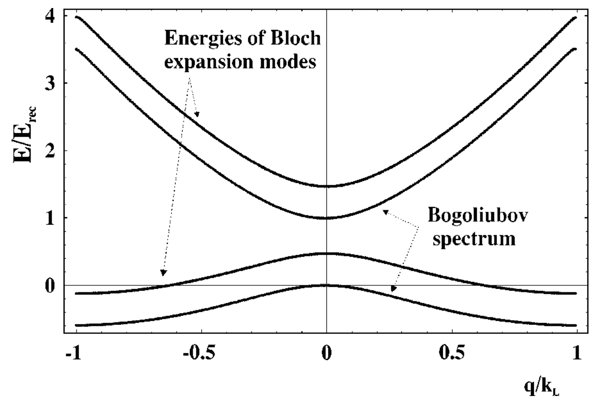

FIG. 2. Real part of the Bogoliubov spectrum and energies of the expansion modes for the first two bands in units of the recoil energy $E_{rec} = \hbar^2 k_L^2/(2M)$.

IV. ONE-DIMENSIONAL RESULTS

In one dimension the convergence of the SCF algorithm can easily be achieved. We have used up to 180 Bloch modes to expand the exact soliton wave function which covered the upper half of the lowest-energy band and the lower third of the second Bloch energy band. However about 40 modes covering the effective-mass region around the upper band edge were sufficient to get about the same numerical accuracy. The (real) expansion coefficients for the gap soliton can be seen in Fig. 1. It is interesting to note that the SCF method essentially amounts to removing that part of the approximate solution (3) which corresponds to the second energy band. The exact numerical solution therefore is indeed centered around the upper band edge of the first energy band, where the effective mass is approximately constant. We have checked whether the final result of the SCF algorithm indeed describes a soliton by using it as an initial condition for the split-step direct integration of the GPE. It was found that the solution does not change its shape for times exceeding 0.2 s. To verify the stability of the 1D gap soliton we have calculated the quasiparticle spectrum for up to 350 Bloch modes as basis functions. The result for the real part of the spectrum can be seen in Fig. 2 together with the Bloch mode energies. As one can see the two spectra are very similar apart from a constant shift. This shift is given by the chemical potential and arises because of the corresponding phase factor in ansatz (4). The similarity of the curves is due to the small number of condensed atoms ($N_A = 400$) and the corresponding small collision effects. With a numerical accuracy of 10^{-15}, the imaginary part of the quasiparticle spectrum is zero except for a single mode (and the corresponding mode with complex conjugated frequency) for which the frequency is purely imaginary. The value of the imaginary part for this mode depends on the degree of convergence of the SCF wave function and on the number of modes included. For the analytical, nonconverged solution (3) the value of the imaginary part of the mode is $\mathrm{Im}(\omega_U) \approx 0.02 \times \omega_\perp$, where $\omega_\perp$ is the transverse trap frequency. For a well-converged solution it is always very small, $\mathrm{Im}(\omega_U) < 0.001 \times \omega_\perp$.

To understand the unstable mode better a few general facts about the quasiparticle spectrum are helpful: for any potential and any stationary solution of the GPE the mode $(u,v) = (\psi_0, \psi_0^*)$ is a quasiparticle mode with frequency $\omega = 0$. This mode is a Goldstone mode associated with the symmetry of the energy functional

$$E = \int \left\{ \psi_0^* \left(\frac{\mathbf{p}^2}{2M} + V \right) \psi_0 + \kappa |\psi_0|^4 \right\} d^n x \tag{8}$$

with respect to a global phase change $\psi' = \exp(i\alpha)\psi$. If the external potential V is absent then, for any stationary solution ψ_0 of the GPE, there is a second Goldstone mode $(u,v) = (\nabla \psi_0, -\nabla \psi_0^*)$. It arises because of the invariance of E against spatial translations. If the effective mass approximation was exact then the gap soliton would fulfill the same equation as the bright soliton does in free space. Consequently, it would also possess a translational Goldstone mode. However, since the periodic optical potential does explicitly break the translational invariance one can expect a shift of the complex Goldstone frequency away from zero. A second effect that explicitly breaks translational invariance is the finite number of Bloch basis functions used in the numerical calculations. This is equivalent to an optical lattice placed in a box whose length is a finite multiple of the lattice period. In our case, the box contained 260 periods.

That the unstable mode indeed corresponds to a modified Goldstone mode can also be seen by looking at its expansion coefficients (Fig. 3). Obviously the shape of $u(q)$ is approximately given by $q\psi_0(q)$ which would describe the Goldstone mode if the quasimomentum q is replaced by the real momentum as it is done in the effective-mass approximation. The question remains whether this tiny instability associated with $\mathrm{Im}(\omega_U) < 0.001 \times \omega_\perp \approx 0.1\ \mathrm{s}^{-1}$ results from numerical aberrations, from the finite number of periods, or whether it is a real physical effect. To shed some light on this question we also have performed a stability analysis of the bright soliton in free space using the same algorithm. We found a similar behavior: a Goldstone mode develops an imaginary

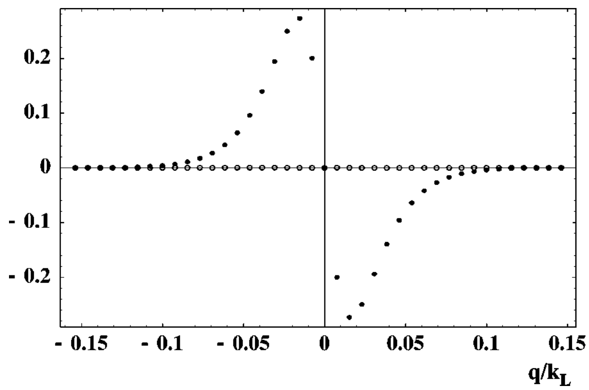

FIG. 3. Expansion coefficients for the modified translational Goldstone mode. Shown is the real part (dots) and imaginary part (circles) of the function $u(z)$. Numerically it was found that $v(z) \approx -u(z)$.

eigenvalue, but this time it is the mode associated with the phase transformation. Since one can prove that this mode has zero frequency, we conclude that the tiny imaginary eigenvalue for Goldstone modes is a spurious numerical effect and the gap soliton is dynamically stable in one dimension.

V. RESULTS IN TWO AND THREE DIMENSIONS

Having examined the stability of the 1D gap soliton it is of interest whether it will remain stable in a quasi-one-dimensional situation. The condition for the latter is usually formulated as follows: the interaction energy, which leads to a coupling between different modes of the corresponding linear Schrödinger equation, should be much smaller than the excitation energy of the transverse trapping potential. If this is fulfilled, a BEC in its ground state will effectively behave like a one-dimensional quantum gas since transverse excitations are off-resonant and therefore suppressed.

However, this is not the case for a gap soliton. The reason can be seen by looking at Fig. 4 which displays the mode energies of noninteracting atoms in the optical lattice and with a tight harmonic transverse confinement around the upper band edge. The solid line displays the energy of atoms in the transverse ground state and with longitudinal quasimomentum q in the lowest-energy band. Each dashed line corresponds to transversally excited atoms with a transverse energy of $2\hbar\omega_\perp$ to $6\hbar\omega_\perp$, respectively. It is important to observe that there are resonances between transversally unexcited atoms with $q=0$ and transversally excited atoms with $q\neq 0$. Since the gap soliton is a superposition of Bloch modes around the upper band edge $(q=0)$, these resonances have the consequence that even for tight transverse confinement a true gap soliton does not exist. This situation is qualitatively the same in two and in three dimensions since in both cases the free-energy levels are given by those of Fig. 4. Since in two dimensions the transverse trapping potential is one dimensional, the multiplicity of the energy levels is always one. This is different in three dimensions where a transverse excitation energy of $n\hbar\omega_\perp$ has an $(n+1)$-fold degeneracy. The number of resonant states is therefore larger than in two dimensions. Although a true gap soliton does not exist it is of interest to examine a *quasigap soliton* of the form

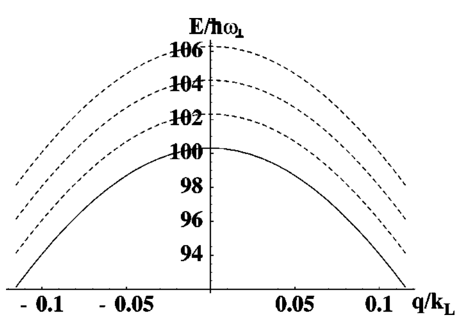

FIG. 4. Energy eigenvalues around the upper band edge for noninteracting atoms in an optical lattice and with a transverse trapping potential. Due to resonances between longitudinal and transverse excitations the gap soliton will be unstable against transverse decay. The physical parameters are given in the text.

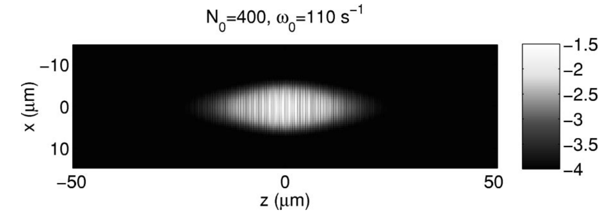

FIG. 5. Density of the initial state wave function $\log_{10}|\psi|^2$ for 400 atoms in the BEC.

$$\psi_{\mathrm{quasi}} = \psi_{\mathrm{gap}}(z,t)\,\varphi_0(x,y), \tag{9}$$

with φ_0 denoting the transverse ground state. It should be possible to produce a state like ψ_{quasi} using dispersion management [18]. Though the quasi gap soliton is not a true stationary solution of the GPE, it may be sufficiently stable to allow for experimental observation. A signature of a quasigap soliton would be a strongly suppressed dispersion of the wave packet along the z axis. To analyze the time scale on which this state decays, we first note that the transverse ground state has even parity and because of parity conservation can only couple to even excited levels $2n\hbar\omega_\perp$. This is the reason why we omitted odd transverse excitations in Fig. 4.

We have used the two-dimensional version of state (9) as shown in Fig. 5 as initial condition and numerically studied the time evolution of it. To study the influence of the transverse confinement we have considered three BECs with 400, 1600, and 25 600 atoms in Figs. 6(a), 6(b), and 6(c), respectively. The number of atoms N_A and the transverse confinement frequency $\omega_\perp$ have been simultaneously varied keeping the product $\omega_\perp N_A$ constant. Consequently the interaction energy in Eq. (8) is kept constant, since it is proportional to $(\omega_\perp N_A)^2$. This also ensures that the first-order soliton condition is fulfilled. The result of the numerical time evolution after 23 ms is shown in Fig. 6. While some excitations are observable, the state still looks very much like the quasigap soliton in Fig. 5. This situation changes after 0.23 s (Fig. 7). On Figs. 7(b) and 7(c) one can see strong excitations which are growing with decreasing transverse excitation frequency, whereas the state shown on Fig. 7(a) still resembles the gap

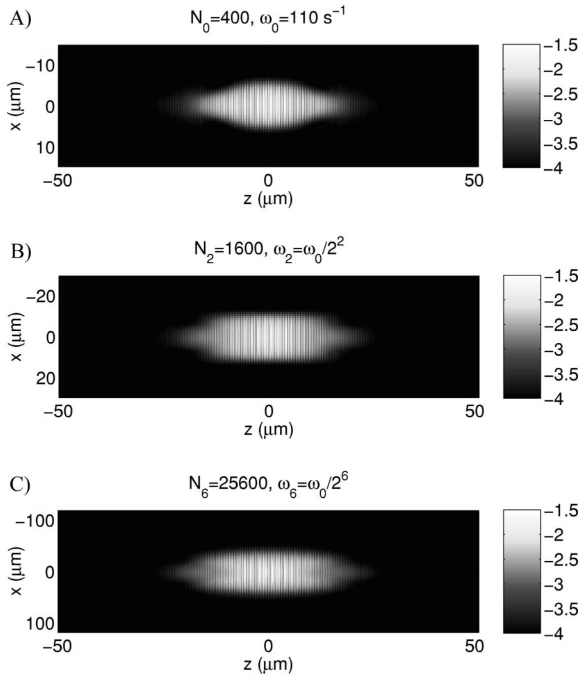

FIG. 6. Density of the wave function $\log_{10}|\psi|^2$ at $t=23$ ms for (a) 400 atoms, (b) 1600 atoms, and (c) 25 600 atoms in the BEC. See text for details.

soliton state on Fig. 5 quite well. This is a reasonable result since the ratio of the interaction energy and the transverse excitation energy is greater than one for Figs. 7(b) and 7(c) hence the nonlinear coupling is strong enough to excite the transverse modes. The figure suggests that in particular the transverse modes with $2\hbar\omega_\perp$ energy are strongly excited because for a fixed value of z there are three density maxima in the x direction. This is in agreement with the qualitative predictions which we have made above. To verify this result we also have calculated the quasiparticle spectrum of state (9) in two and three dimensions. In principle, since the quasigap soliton is not a stationary state, the spectrum is not enough to predict the evolution of it accurately and a more sophisticated approach is needed [17]. However, to gain a qualitative understanding of the time scale on which $\psi_{\rm quasi}$ decays the imaginary part of the spectrum is sufficient.

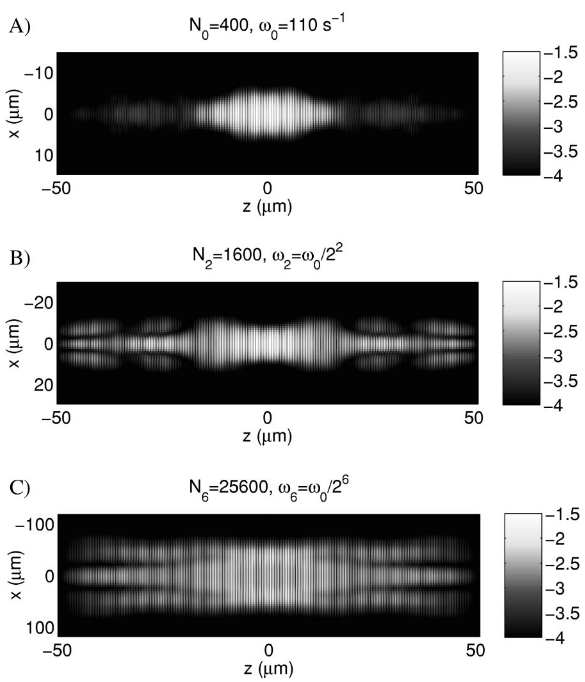

FIG. 7. Density of the wave function $\log_{10}|\psi|^2$ at $t=0.23$ s for (a) 400 atoms, (b) 1600 atoms, and (c) 25 600 atoms in the BEC. See text for details.

In 2D we used up to 100 Bloch wave functions and up to 19 one-dimensional eigenstates of the transverse harmonic trap as basis modes. The quasigap soliton was expanded using 51 Bloch states. It turned out that there are generally quite many unstable modes, but only few of them do have a considerable overlap with the collective wave function. The number of unstable modes depends on the number of basis states used to expand the BDGE since the number of resonant transversely excited states is growing. However, it turned out that this basis dependence does only affect modes with a small instability. Some examples are displayed in Fig. 8. Mode A of Fig. 8 corresponds very well to the anticipated resonant excitation of transverse modes. It has a nonzero overlap with the quasigap soliton and otherwise only populates even transversely excited basis modes. Correspondingly its instability is rather large; only the instability of mode B of Fig. 8, which roughly describes the phase Goldstone mode, decays faster. Mode B of Fig. 8 is unstable because, as described above, the quasigap soliton is not a stationary solution of the GPE. However, due to a coupling to transversely excited states the decay rate is strongly enhanced as compared to the nonconverged one-dimensional quasigap soliton (the modulus of the transversely excited mode coefficients is less than 0.1 and is therefore not visible in Fig. 8. Mode C of Fig. 8 is typical for the many unstable modes which have no overlap with the quasigap soliton. Therefore, unless transversal excitations are created during the experimental preparation of the quasigap soliton, they do not contribute to the decay of it. These modes appear because the collective wave function provides a linear coupling term between transversely excited states which are in resonance with each other. Such modes can exist for even and odd transverse excitation levels without violating parity. Finally, mode D of Fig. 8 describes a somewhat off-resonant coupling between the transverse ground state and transversely excited states. Because of its off-resonant nature its decay rate $\mathrm{Im}(\omega)$ is considerably smaller than for the resonant mode A of Fig. 8. To analyze the dynamical instability of a 3D quasigap soliton we used again Bloch wave functions for the longitudinal expansion of the Bogoliubov modes. For the two transversal directions we have chosen a basis of states which are both eigenstates of the Hamiltonian and the angular momentum operator L_z. These states can be constructed by using the creation operators $c_\pm := (a_x^\dagger \pm i a_y^\dagger)/\sqrt{2}$ [19]. The basis states are then given by

$$|n,m\rangle := \frac{(c_+^\dagger)^{(n-m)/2}(c_-^\dagger)^{(n+m)/2}}{\sqrt{[(n+m)/2]![(n-m)/2]!}}|0\rangle. \qquad (10)$$

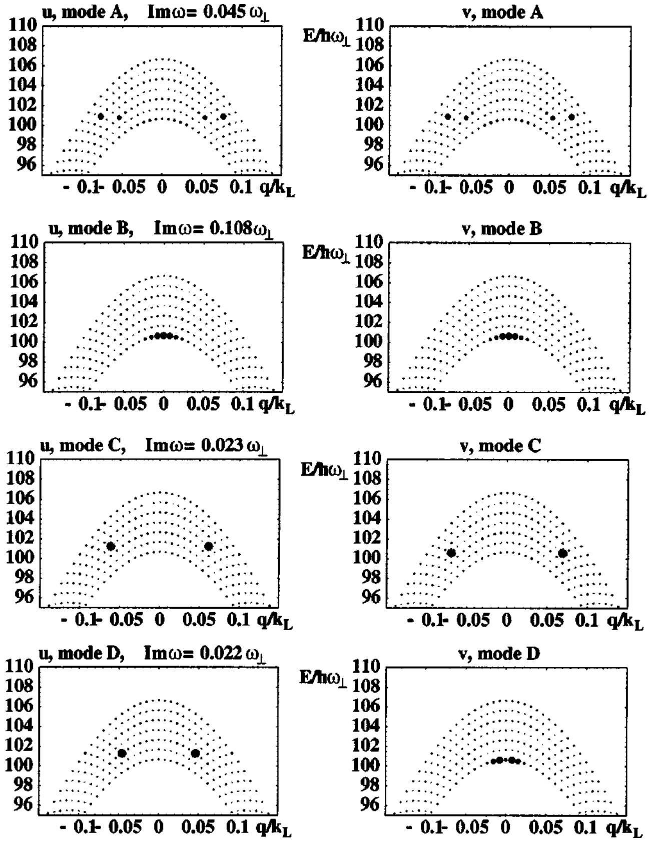

FIG. 8. Selected unstable Bogoliubov modes for a 2D quasigap soliton. Shown is the energy E (in units of $\hbar\omega_\perp$) of the basis functions as a function of the quasimomentum q (in units of k_L). The lowest parabola corresponds to the transverse ground state. The other parabolas describe a transverse excitation of $n\hbar\omega_\perp, n=1,2,\ldots$. The thickness of the dots corresponds to the modulus of the u or v coefficients with respect to the corresponding basis function. The smallest (largest) dots correspond to coefficients of modulus between 0 and 0.1 (0.5), respectively. The imaginary part $\mathrm{Im}(\omega)$ of the unstable modes is given in units of the transverse trap frequency $\omega_\perp = 110\ \mathrm{s}^{-1}$.

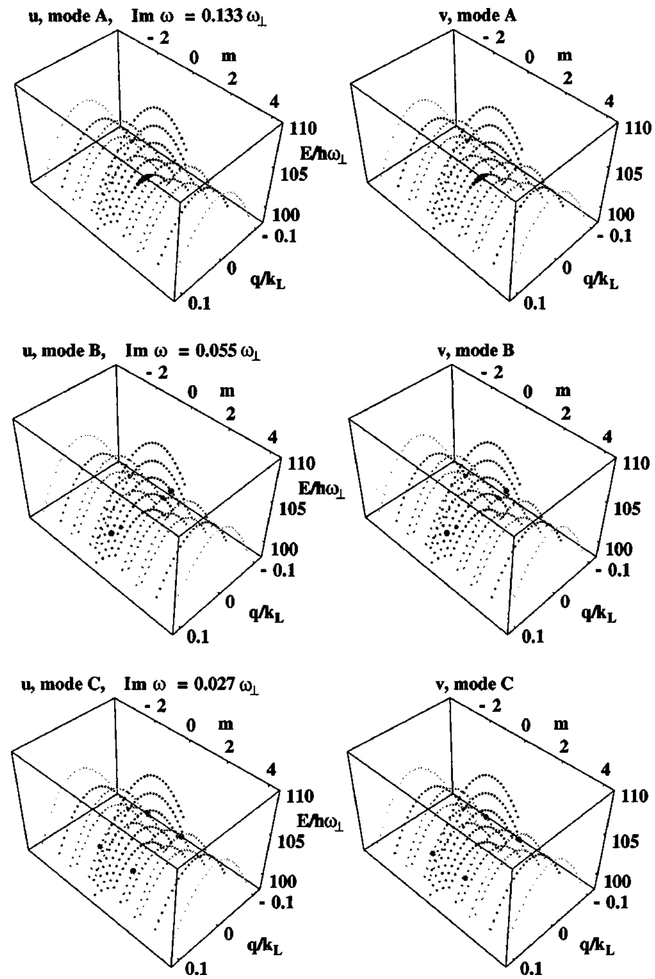

FIG. 9. Selected unstable Bogoliubov modes for a 3D quasigap soliton. The units are as in Fig. 8, but in addition to the basis energy E and quasimomentum q, a third quantum number m is introduced which is associated with the axial angular momentum $L_z = m\hbar$ of the basis functions.

The energy of these states is given by $n\hbar\omega_\perp, n=0,1,2,\ldots$ and their angular momentum by $m\hbar, m=-n,-(n-2),\ldots,n$. We used again 51 Bloch states to expand the Bogoliubov modes along the z axis and up to 30 transverse modes as given in Eq. (10). We found again a large number of unstable modes which spuriously depends on the number of transverse basis states. However, similarly to the 2D case, modes with an instability $\mathrm{Im}(\omega) > 0.02\omega_\perp$ do not depend significantly on the number of basis states. Some of these unstable Bogoliubov modes in 3D are shown in Fig. 9. Mode A of Fig. 9 is again roughly the phase Goldstone mode plus some small transversal excitations not visible in the figure. Due to the larger number of nearly resonant basis states the decay rate is even larger than in the 2D case. Mode B of Fig. 9 is again a superposition of resonant states and, as in 2D, has the second largest decay rate. These two states as well as most of the other unstable modes are solely composed of states with zero angular momentum, i.e., states which are symmetric in the x-y plane. They are therefore very similar to the 2D case. A new kind of instability in 3D occurs in Mode C Fig. 9, which has no overlap with the quasigap soliton. It is composed out of states with $m=\pm 1$ and thus describes the decay or increase of rotating states linearly coupled by the collective wave function. We found 80 unstable modes of this kind.

VI. CONCLUSION

We have examined the dynamical instability of gap solitons in a BEC in one, two, and three spatial dimensions under the condition of tight transversal confinement. Using different methods we found that a truly one-dimensional gap soliton is stable. In higher-dimensions transversal excitations which are resonant with the upper band edge forbid the existence of a real gap soliton. However, a quasigap soliton may be experimentally prepared which behaves like the 1D gap soliton for a time smaller than the smallest decay time of one of the unstable Bogoliubov modes. In 3D we numerically found this time to be in the order of $1/(0.133\times\omega_\perp)$ which is the decay time of mode A in Fig. 9. For a transverse trap frequency of $\omega_{\perp} = 110\ \mathrm{s}^{-1}$ we expect the quasigap soliton to be sufficiently stable for about 70 ms. This should be long enough for experimental observation.

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